Part I. Do three of these problems.

I.1 Let $G$ be a group. Assume that $G$ is generated by two subgroups, $N$ and $A$, such that $A$ is abelian and $N$ is finite and normal in $G$. Show that the center $Z(G)$ has finite index in $G$.

I.2 Let $R$ be a unique factorization domain and let $P$ be a nonzero prime ideal of $R$. Show that the following are equivalent:

(i) there is no prime ideal $Q$ with $0 \subseteq Q \subsetneq P$;

(ii) $P$ is principal, that is, $P = Rx$ for some $x \in R$.

I.3 Let $V = \mathbb{R}^n$, where $n$ is a positive integer, and let $v \in V$ be such that $v \cdot v = 2$, where $\cdot$ is the ordinary dot product.

(a) Show that, defining $s(x) = x - (x \cdot v)v$ for $x \in V$, one obtains a linear transformation $s = s_v \in \text{End}_\mathbb{R}(V)$ satisfying $s^2 = \text{Id}_V$ and $\text{rank}(s - \text{Id}_V) = 1$.

(b) Assume that $t \in \text{End}_\mathbb{R}(V)$ satisfies $t(v) = -v$, $t$ induces the identity on $V/\mathbb{R}v$, and the composite $f = s \circ t$ satisfies $f^m = \text{Id}_V$ for some positive integer $m$. Show that $t = s$.

I.4 Let $R$ be a commutative ring. Recall that an element $r \in R$ is said to be nilpotent if $r^n = 0$ for some integer $n \geq 0$. Prove that:

(a) The set $N = \{\text{all nilpotent elements of } R\}$ is an ideal of $R$ that is contained in every prime ideal of $R$.

(b) A polynomial $f = r_0 + r_1x + \cdots + r_dx^d \in R[x]$ is nilpotent if and only if all coefficients $r_i \in R$ are nilpotent.
Part II. Do two of these problems.

II.1 Let $F_n$ be the group freely generated by symbols $a_1, \ldots, a_n$ and let $B_n$ be the group with the presentation

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1, \ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ \forall \ i \rangle$$

Prove that the formulas

$$\Psi(\sigma_i)(a_j) = \begin{cases} a_i a_{i+1} a_i^{-1} & \text{if } j = i \\ a_i & \text{if } j = i + 1 \\ a_j & \text{otherwise} \end{cases}$$

define a group homomorphism $\Psi : B_n \to \text{Aut}(F_n)$, where $\text{Aut}(F_n)$ denotes the automorphism group of $F_n$. Prove that, for every $1 \leq i \leq n - 1$, the automorphism $\Psi(\sigma_i)$ has an infinite order.

II.2 Let $f(x) \in F[x]$ be a (non-constant) polynomial, where $F$ is any field. Define the discriminant $\delta$ of $f(x)$ and prove that $\delta \in F$.

II.3 (a) Let $p(x)$ be a cubic irreducible polynomial in $\mathbb{Q}[x]$, $\delta$ be its discriminant and $G$ be its Galois group. Prove that $G$ is isomorphic to $A_3$ if and only if $\delta = q^2$ for some $q \in \mathbb{Q}$. Prove that, otherwise, $G \cong S_3$.

(b) Let $p(x) \in \mathbb{Q}[x]$ be an irreducible polynomial with at least one real root and at least one complex root with a non-zero imaginary part. Prove that the Galois group of $p(x)$ is non-Abelian.