Part I. Do three of these problems.

I.1 Let $G$ be a group of order $p^n$ ($p$ is prime). Prove that $G$ has a normal subgroup of order $p^m$ for all $0 \leq m \leq n$.

I.2 Recall that a commutative ring $R$ with identity ($1 \neq 0$) is called local if it has exactly one maximal ideal.
(a) Prove that a commutative ring $R$ with identity is local if and only if all non-units of $R$ form an ideal of $R$; this is exactly the unique maximal ideal of $R$.
(b) Prove that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to a direct product of local rings for every $0 \neq n \in \mathbb{Z}$.

I.3 Let $F$ be a field of characteristic 0 and let $V = \bigoplus_{i=0}^{n} F x^i$ be the $F$-vector space of all polynomials of degree at most $n$. Let $D$ be the endomorphism of $V$ that is given by formal differentiation, $\frac{d}{dx}$.
(a) Find the Jordan canonical form of $D$.
(b) Determine all $D$-invariant subspaces of $V$.

I.4 Let $F$ be a field and $n$ a positive integer. Consider the polynomial ring $F[x_1, \ldots, x_n]$ and the ring $R$ of all functions $F^n \to F$, with pointwise addition and multiplication of functions. Show that the evaluation map
\[ \phi: F[x_1, \ldots, x_n] \to R, \quad \phi(f)(\lambda_1, \ldots, \lambda_n) = f(\lambda_1, \ldots, \lambda_n) \]
is injective if and only if $F$ is infinite.
II.1 Let $\mathbb{F}_p$ be the finite field with $p$ elements. A *maximal flag* in the $\mathbb{F}_p$-vector space $V = \mathbb{F}_p^n$ is a sequence of subspaces,

$$V = V_n \supset V_{n-1} \supset \cdots \supset V_2 \supset V_1 \supset V_0 = \{0\},$$

where $\dim V_k = k$. Let $U$ be the subgroup of $\text{GL}_n(\mathbb{F}_p)$ which consist of elements $g$ satisfying

- $g(V_k) = V_k$, and
- $g$ induces the identity map on $V_k/V_{k-1}$

for all $n \geq k \geq 1$. Prove:

(a) $U$ is a Sylow $p$-subgroup for every maximal flag.

(b) Every Sylow $p$-subgroup of $\text{GL}_n(\mathbb{F}_p)$ is of this form.

(c) The number of Sylow $p$-subgroups of $\text{GL}_n(\mathbb{F}_p)$ is given by

$$n_p(\text{GL}_n(\mathbb{F}_p)) = (1 + p)(1 + p + p^2) \ldots (1 + p + p^2 + \cdots + p^{n-1}).$$

II.2 Let $V$ be a finite-dimensional vector space over the algebraically closed field $F$. Recall that an endomorphism $\phi \in \text{End}_F(V)$ is called *diagonalizable* if $V$ has a basis consisting of eigenvectors for $\phi$. Prove:

(a) $\phi$ is diagonalizable if and only if the minimal polynomial $m_\phi(t) \in F[t]$ is separable.

(b) If $W \subseteq V$ is a subspace such that $\phi(W) \subseteq W$, then the restriction $\phi|_W \in \text{End}_F(W)$ is diagonalizable.

II.3 Let $\zeta = e^{2\pi i/7} \in \mathbb{C}$. Determine the degree of the following elements over $\mathbb{Q}$.

(a) $\zeta + \zeta^5$,

(b) $\zeta^3 + \zeta^5$,

(c) $\zeta^3 + \zeta^5 + \zeta^6$. 

2