PART I: Do three of the following problems.

1. Given an additive abelian group \( A \) and a positive integer \( m \), set \( mA = \{ ma : a \in A \} \). Now let \( A \) be a finitely generated but not finite additive abelian group. Prove that there exists a positive integer \( n \) such that \( nA \) is a nonzero free abelian group.

2. Let \( n \) be a positive integer, and let \( N \) be an \( n \times n \) complex matrix. Suppose for every \( n \times n \) complex matrix \( A \) there exists a complex \( n \times n \) matrix \( B \) such that \( AN = NB \). Prove that \( N \) is either the zero matrix or is invertible.

3. Let \( K \) be a field. Prove that the polynomial ring in two variables \( K[x, y] \) is not a principal ideal domain.

4. Let \( F \) be a subfield of \( \mathbb{C} \). Suppose that \( [F : \mathbb{Q}] \) is an odd positive integer and that \( F \) is a normal extension of \( \mathbb{Q} \). Prove that \( F \) is contained in \( \mathbb{R} \).
Part II: Do two of the following problems.

1. Let $G$ be a finite group, and let $P$ be a Sylow $p$-subgroup of $G$. Let $H$ be a subgroup of $G$, and let $N$ be a normal subgroup of $G$.

   (a) Prove that $gPg^{-1} \cap H$ is a Sylow $p$-subgroup of $H$ for some $g \in G$.

   (b) Prove that $P \cap N$ is a Sylow $p$-subgroup of $N$.

   (c) Prove that $PN/N$ is a Sylow $p$-subgroup of $G/N$.

2. Let $R$ be a ring with identity and suppose that $R$ contains a unique maximal left ideal $M$.

   (a) Prove that $Ma \subseteq M$ for all $a \in R$, and conclude that $M$ is a two-sided ideal of $R$.

   (b) Prove that $M$ is equal to the set of non-invertible elements of $R$. (Recall that an element $u$ of $R$ will be invertible if and only if there exists an element $v$ of $R$ such that $uv = vu = 1$.)

   (c) Prove that $M$ is also the unique maximal right ideal of $R$.

3. Let $K$ be the splitting field over $\mathbb{Q}$, in $\mathbb{C}$, of $x^4 - 2$. Let $G = \text{Gal}(K/\mathbb{Q})$.

   (a) Determine the order of $G$, and show that $G$ is isomorphic to the group of symmetries of a plane geometric figure.

   (b) Specify the subfields of $K$. For each subfield $F$ of $K$, give field generators over $\mathbb{Q}$, and give the degree $[F : \mathbb{Q}]$. 