Comprehensive Examination January, 1999

ALGEBRA

Part I: Do three of the following problems.

1. R is a ring with 1. An element $x \in R$ is called nilpotent if $x^k = 0$ for some k. Let $N = N(R) = \{\text{nilpotent elements of } R\}$.

   (a) Show that if R is commutative then N is an ideal. Moreover, R/N has no nonzero nilpotent elements.

   (b) Find N explicitly for $R = \mathbb{Z}/900\mathbb{Z}$ and $R = \mathbb{Q}[x]/(x^5 - x^4)$.

2. Let S_n denote the symmetric group on n symbols and let $\sigma \in S_n$ be an n-cycle. Show that the centralizer of σ in S_n is precisely $\langle \sigma \rangle$.

3. Let A be an $n \times n$-matrix over a field F. Prove:

   (a) If A is nilpotent (i.e., some power of A is the zero matrix) then $\text{trace}(A^m) = 0$ holds for all $m \geq 1$.

   (b) For $n = 2$ and F of characteristic $\neq 2$, prove the converse: If $\text{trace}(A) = 0 = \text{trace}(A^2)$ then A is nilpotent.

4. Let R be a commutative ring (with 1) and let $F \subseteq R$ be a subring. Assume that F is a field and that $\dim_F R < \infty$. Show that R either has zero divisors (that is, nonzero elements whose product is zero) or is a field.

Part II: Do two of the following problems.

1. Let R be a commutative ring with 1, and let $f(x) = r_0 + r_1 x + \ldots r_n x^n \in R[x]$. Prove:

   (a) If r_0 is invertible in R and all r_i $(i > 0)$ are nilpotent then $f(x)$ is invertible in $R[x]$.

   (b) If $f(x)$ is invertible in $R[x]$ then r_0 is invertible in R.

   (c) If $f(x)$ is invertible in $R[x]$ then all r_i $(i > 0)$ are nilpotent.

2. Let p be a prime number. Determine the number of Sylow p-subgroups of the symmetric group S_p.

3. Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{C}$. Determine:

   (a) The minimal polynomial $f(x)$ of α over $\mathbb{Q}$.

   (b) The splitting field of $f(x)$ over $\mathbb{Q}$.

   (c) The Galois group of $f(x)$ over $\mathbb{Q}$.