PhD Algebra Exam
Spring 94

Part I: Do three of these problems.

1. A group G is called a p-group (for p a prime) if the order of G is a power of p.

   a) Show that if G is a p-group and A is a normal subgroup G of order p, then A is contained in the center of G.
   a) Give examples to show that this does not hold if $|A| = p^2$, or if $A| = p$ and G is not a p-group.

2. Let T be a linear transformation on a finite dimensional vector space V, such that $T^2 = T$.

   a) Show that $V = T(V) \oplus \ker T$
   b) What is the matrix of T with respect to a basis chosen according to this direct sum decompostion (*i.e.*, conjunction of basis of $T(V)$ with basis of $\ker T$)?
   c) Compute such a basis for $T = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$ acting on $\mathbb{R}^2$.

3. Recall the following definitions: For a group G, subgroup $H \leq G$, we say H is a characteristic subgroup if every automorphism of G sends H into itself. We denote $H_1 = [H, H] =$ commutator subgroup of H, and define the "lower central series" $\{G_i\}$ of G by $G_i = [G_{i-1}, G_{i-1}]$.

   a) Show that if H is characteristic in G then H is normal in G.
   b) Show by induction that the lower central series subgroups are all characteristic in G.
   c) Compute the lower central series for the two non-abelian groups of order 8.

4. Identify the splitting field of the polynomial $f(x) = x^3 - 2$ over each of the following fields: a) $\mathbb{Z}_2$ b) $\mathbb{Z}_3$ c) $\mathbb{Z}_5$ d) $\mathbb{Z}_7$

Part II: Do two of these problems.

5. Let S and T be linear transformations on a finite dimensional vector space V.

   a) Suppose v is an eigenvector for both S and T. Show v is also an eigenvector for $S + T$ and for ST. What is the relationship between the corresponding eigenvalues?
   b) Suppose λ is a non-zero eigenvalue of AB. Show that λ is also an eigenvalue of BA. What is the relationship between the corresponding eigenvectors?

6.

   a) Show that $\mathbb{Z}[i]$ is a Euclidean ring.
   b) Show that $\mathbb{Z}[\sqrt{-5}]$ is not a Euclidean ring.

7. Let ζ be a primitive 16^{th} root of unity (so $\zeta^{16} = 1$) over the rationals $\mathbb{Q}$.

   a) Find the irreducible polynomial for ζ over $\mathbb{Q}$.
   b) Identify the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
   c) How many subfields does $\mathbb{Q}(\zeta)$ have which are quadratic over $\mathbb{Q}$?

Typeset by $\mathcal{AMS}$-TEX