PhD Algebra Exam
Spring 1989

Part I: Do three of these problems.

1. Let $A$ be the real $3 \times 3$ matrix all of whose entries are 1:

$$
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
$$

Find

a) the eigenvalues of $A$

b) for each eigenvalue, a basis for the space of eigen vectors

c) the characteristic polynomial of $A$

d) the minimal polynomial of $A$

e) the Jordan normal form of $A$

2. Let $\mathbb{Z}_m$ and $\mathbb{Z}_n$ be the cyclic groups of orders $m$ and $n$.

a) Prove that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if $\text{GCD}(m, n) = 1$.

b) Prove that every subgroup of a cyclic group is cyclic.

3. Let $R$ be an associative ring with identity such that every element is idempotent; that is, $x^2 = x$ for all elements $x \in R$.

a) Prove that $R$ is commutative and has characteristic 2.

b) Give two examples of such rings, one finite and one infinite.

4. True or false: Justify if true, give counterexample if false.

a) An algebraic extension of a field has finite degree.

b) A solvable group is abelian.

c) A unique factorization domain is a principal ideal domain.

d) An infinite field has characteristic zero.

e) If a group is abelian then every subgroup is normal.

Part II: Do two of these problems.

5. Let $f(x)$ be an irreducible cubic polynomial over the rationals $\mathbb{Q}$ with at least one non-real root. Let $\mathbb{K}$ be the splitting field of $f(x)$.

a) Show $[\mathbb{K} : \mathbb{Q}] = 6$

b) Show that the Galois group $G(\mathbb{K}/\mathbb{Q})$ is isomorphic to the symmetric group $S_3$.

c) Show that there exist irreducible cubics over $\mathbb{Q}$ whose Galois groups are not isomorphic to $S_3$, and say what the group must be.

6. Let $A$ be an invertible matrix over a finite field $\mathbb{F}$.

a) Show that there is an integer $k$ such that $A^k = I$ (identity).

b) Suppose the characteristic of $\mathbb{F}$ is $p$, and let $a \neq 0$ be an element of $\mathbb{F}$. Find a value of $k$ which works for the matrix

$$
A = \begin{bmatrix}
1 & a \\
0 & 1
\end{bmatrix}
$$
c) Find a value of $k$ which works for the matrix

$$
A = \begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}
$$

7. Let $p$ and $q$ be primes, not necessarily distinct. Prove that any group of order $p^2q$ is solvable; consider separately the cases $p = q$ and $p \neq q$. (You may assume Sylow theory and the class equation.)