Part I. Do three of these problems:

I.1 Consider the ring \( \mathbb{Z}[x] \). Give an example of a non-zero prime ideal in \( \mathbb{Z}[x] \) that is not maximal. Now let \( R \) be an arbitrary principal ideal domain. Prove that every non-zero prime ideal in \( R \) is also maximal.

I.2 Let \( p \) be prime and let \( A \) be a \((p - 1) \times (p - 1)\) matrix with rational entries such that \( A \neq I_{p-1} \), the \((p - 1) \times (p - 1)\) identity matrix, but

\[
A^p = I_{p-1}.
\]

Find the rational canonical form of \( A \) over \( \mathbb{Q} \) and the Jordan canonical form of \( A \) over \( \mathbb{C} \).

I.3 Let \( R \) be an integral domain. Prove that the polynomial ring \( R[x] \) is also an integral domain. Prove that the units of \( R[x] \) are precisely the units of \( R \). Give an example of a commutative ring \( R \) (with \( 1 \neq 0 \)) and a pair of polynomials \( p(x), q(x) \in R[x] \) of positive degrees for which

\[
p(x) \cdot q(x) = 1.
\]

I.4 Find all isomorphism classes of groups \( G \) of order 12 which satisfy this property: the Sylow 3-subgroup \( H \) of \( G \) is normal in \( G \).
Part II. Do two of these problems:

II.1 Let $R$ be a principal ideal domain. Prove directly (i.e., without invoking Hilbert’s Basis Theorem) that every ideal in the polynomial ring $R[x]$ is finitely generated.

Hint: Let $I$ be a non-zero ideal of $R[x]$. Prove that the union

$$L = \{0\} \cup \{ \text{leading coefficients of non-zero polynomials in } I \}$$

is an ideal of $R$. Since $R$ is a principal ideal domain, $L = (r)$ for some $r \in R$ and there is a polynomial $f(x) \in I$ whose leading coefficient is $r$. Next, consider polynomials in $I$ of degrees $< \deg f(x)$.

II.2 Let $K \supset F$ be a finite separable field extension. Let us denote by $\text{Emb}(K/F)$ the set of all ring homomorphisms

$$\psi : K \to \overline{F},$$

such that $\psi(a) = a$ for every $a \in F$, where $\overline{F}$ is a fixed algebraic closure of $F$. Prove that the number of elements in $\text{Emb}(K/F)$ is precisely $[K : F]$.

II.3 Let $G$ be a group having a series of subgroups $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that all $G_i$ are normal in $G$ and all factors $G_i/G_{i-1}$ are cyclic. (Such a group $G$ is called supersolvable.) Show that $G$ contains a nilpotent normal subgroup of finite index. Give an example of a non-nilpotent supersolvable group.