Part I. Do three of these problems.

I.1 Let $G$ be a group (not necessarily finite). For each $g \in G$, recall that the elements $hgh^{-1}$, for $h \in G$, are the conjugates of $g$. Put

$$\mathcal{FC}(G) := \{g \in G \mid g \text{ has only finitely many conjugates}\}.$$  

(a) Prove that $\mathcal{FC}(G)$ is a characteristic subgroup of $G$. (Recall that a subgroup $H \leq G$ is characteristic if $H$ is mapped to itself under all automorphisms of $G$.)

(b) Show that all finite normal subgroups of $G$ are contained in $\mathcal{FC}(G)$.

(c) Consider the infinite dihedral group $D_\infty = \langle x,y \mid y^2 = 1, yx = x^{-1}y \rangle$. Prove that $y \notin \mathcal{FC}(D_\infty)$. Conclude that an arbitrary finite subgroup of a group $G$ need not be contained in $\mathcal{FC}(G)$.

I.2 Let $V$ be a vector space over a field $F$ and let $\varphi$ be an $F$-linear operator on $V$. Prove there exists an $F$-linear operator $\psi$ on $V$ satisfying $\varphi \psi \varphi = \varphi$, by completing the following steps:

(a) If $C$ is a complement of $\ker(\varphi)$ in $V$ then we have an isomorphism $\varphi\mid_C : C \xrightarrow{\sim} \operatorname{Im}(\varphi)$.

(b) If an $F$-linear operator $\psi$ on $V$ satisfies $\psi\mid_{\operatorname{Im}(\varphi)} = (\varphi\mid_C)^{-1}$ then $\varphi \psi \varphi = \varphi$.

I.3 Let $R$ be a commutative ring (with identity $1 \neq 0$). Assume further that $P_1, \ldots, P_n$ are finitely many prime ideals of $R$ such that the intersection $N = P_1 \cap \cdots \cap P_n$ is nilpotent, that is, $N^t = 0$ for some $t \geq 0$. Prove that every prime ideal of $R$ must contain one of the $P_1, \ldots, P_n$.

I.4 Let $F$ be a field of characteristic $p > 0$ and let $f(X) \in F[X]$ be an irreducible polynomial. Show that $f(X)$ is separable (i.e., $f(X)$ has repeated roots) if and only if $f(X) \notin F[X^p]$. 
Part II. Do two of these problems.

II.1 Let $A_n$ denote the alternating group of degree $n \geq 5$.
(a) Show that $A_n$ acts transitively on $\{1, \ldots, n\}$. Conclude that $A_n$ has a subgroup of index $n$.
(b) Show that $A_n$ has no proper subgroup of index $< n$.

II.2 Let $R$ be a ring (with $1 \neq 0$ but not necessarily commutative). Assume that $x, y \in R$ are given such that $xy = 1$. Put $e = yx \in R$.
(a) Show that $e = e^2$ and conclude that $R = eR \oplus (1 - e)R$.
(b) Show that the map $er \mapsto xer$ ($r \in R$) yields an isomorphism of right $R$-modules, $eR \cong R$.
(c) Conclude that if $R$ contains no infinite direct sums of nonzero right ideals then $e = 1$.

II.3 Let $F/K$ be a Galois extension of fields having characteristic $\neq 2$. Assume that $\text{Gal}(F/K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Prove that $F = K(\sqrt{a}, \sqrt{b})$ for some $a, b \in K$. 

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