Part I. Do three of these problems.

I.1 Let $G$ be a group (not necessarily finite) and let $N$ be a normal subgroup of $G$. Prove:
(a) The centralizer $C_G(N) = \{g \in G \mid gn = ng \text{ for all } n \in N\}$ is a normal subgroup of $G$.
(b) If $N$ is finite then $C_G(N)$ has finite index in $G$.
(c) If $N$ is finite and $G/N$ is cyclic then the center $Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$ has finite index in $G$.

I.2 Let $V$ be a finite dimensional vector space over a field $F$ and let $\langle \cdot , \cdot \rangle : V \times V \to F$ be a bilinear form which is not necessarily symmetric. The right and left radicals of $\langle \cdot , \cdot \rangle$ are the subspaces of $V$ that are defined by
$$R = \{v \in V \mid \langle V, v \rangle = 0\} \quad \text{and} \quad L = \{v \in V \mid \langle v, V \rangle = 0\},$$
respectively. (You need not prove that $R$ and $L$ are subspaces of $V$.)
(a) Use the bilinear form $\langle \cdot , \cdot \rangle$ to construct a linear transformation $f$ from $V$ to the dual space $(V/R)^*$ of $V/R$ such that $\text{Ker } f = L$.
(b) Show that $\dim_F L = \dim_F R$, and deduce that the map $f$ is surjective.

I.3 Let $R$ be a commutative ring and let $P$ be a prime ideal of $R$. If $V$ is a (left) $R$-module, put
$$W = \{v \in V \mid av = 0 \text{ for some } a \in R, a \notin P\}.$$
(a) Show that $W$ is an $R$-submodule of $V$.
(b) If $V$ is an irreducible (i.e., simple) $R$-module and $W = 0$, prove that $P$ is a maximal ideal.

I.4 Let $F$ be a field and let $R \subseteq F$ be a subring of $F$. Assume that $F$ is integral over $R$, that is, for every $\alpha \in F$, there is a monic polynomial in $f \in R[x]$ (depending on $\alpha$) such that $f(\alpha) = 0$. Show that $R$ is a field.
Part II. Do two of these problems.

II.1 Let $G$ be a finite group and let $\text{Syl}_p(G)$ denote the set of all Sylow $p$-subgroups of $G$. Assume that $\text{Syl}_p(G)$ has exactly $n$ elements for some prime $p$. Let $S_n$ denote the symmetric group of degree $n$ and consider the group homomorphism $f : G \to S_n$ that is given by the conjugation action of $G$ on $\text{Syl}_p(G)$. Show that, for any $P, Q \in \text{Syl}_p(G)$ with $P \neq Q$, one has $f(P) \neq f(Q)$.

II.2 Let $R$ be a commutative ring. If $a \in R$, we write $\text{ann}(a) = \{ r \in R \mid ar = 0 \}$ for the annihilator of $a$ in $R$; this is an ideal of $R$. (You need not prove this fact.) Ideals of the form $\text{ann}(a)$ for $0 \neq a \in R$ are referred to as annihilator ideals of $R$. We further let $\mathcal{P} \subseteq R$ denote the set of all elements $a \in R$ such that $\text{ann}(a)$ is a prime ideal of $R$.

(a) Suppose there exists an annihilator ideal of $R$, say $Q$, that is maximal within the set of all annihilator ideals of $R$. Prove that $Q$ is a prime ideal. (In particular, $\mathcal{P}$ is not empty in this case.)

(b) If $a \in \mathcal{P}$ and $r \in R$, show that either $ar = 0$ or $\text{ann}(ar) = \text{ann}(a)$.

(c) If $a, b \in \mathcal{P}$ and $ab \neq 0$, prove that $\text{ann}(a) = \text{ann}(b)$.

II.3 Let $F/K$ be an extension of fields. We say that an element $\alpha \in F$ is abelian if the subfield $K(\alpha) \subseteq F$ is a finite Galois extension of $K$ with abelian Galois group $\text{Gal}(K(\alpha)/K)$. Prove that the set of abelian elements of $F$ is a subfield of $F$ containing $K$. 

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