Comprehensive Examination in Algebra
Department of Mathematics, Temple University

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PART I: Do three of the following problems.

1. Let G be a finite (not necessarily abelian) group, and let n be a positive integer relatively prime to $|G|$. Prove that the function

$$G \xrightarrow{g \mapsto g^n} G$$

is surjective. (This surjectivity amounts to saying that every element of G has an "nth root" in G.)

2. Let A be an additive abelian group. Recall that $mA = \{m.a : a \in A\}$ is a subgroup of A, for all integers m. (You do not have to prove this fact). We say that A is *divisible* provided $nA = A$ for all nonzero integers n. Prove that a nonzero, finitely generated, additive abelian group cannot be divisible.

3. Let R be a commutative ring with identity. The *Jacobson radical* of R, denoted $J(R)$, is the intersection of all of the maximal ideals of R.

   (a) Prove that $a \in R$ is contained in $J(R)$ if and only if $1 + ar$ is a unit in R for all $r \in R$.

   (b) Let k be a field. Prove that the Jacobson radical of $k[x_1, \dots, x_n]$ is the zero ideal, where n is a positive integer.

4. (a) Prove, if n is an integer greater than 2, that there exists a field extension of degree n over $\mathbb{Q}$ that is not normal. (Be careful to completely justify your assertions.)

   (b) Give an example (with proof) of a field extension K/F of finite degree that is not separable.

Part II: Do two of the following problems.

1. Let G be a group of order $224 = 7 \cdot 2^5$. Prove that G is not simple. (You may not use Burnside's "$p^a q^b$ Theorem.")

2. Let R be a PID, and let M be a finitely generated, nonzero (left) R-module. Assume further that M is *completely faithful*: For all nonzero ideals I of R, and all nonzero submodules N of M, the submodule

$$I.N = \left\{ \sum_{k=1}^{\ell} a_k.n_k \;\middle|\; a_k \in I, \;\; n_k \in N, \;\; \ell = 1, 2, \ldots \right\}$$

is also nonzero. (You do not have to prove that $I.N$, in the preceding, is a submodule of M.) Prove that M is a free R-module.

3. Let $p_1, \ldots, p_n$ be pairwise distinct prime positive integers, and set

$$F \;=\; \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n}).$$

   (a) Prove that F is a Galois extension of $\mathbb{Q}$.
   (b) Prove that $[F : \mathbb{Q}] = 2^n$.
   (c) Set $G = \mathrm{Gal}(F/\mathbb{Q})$. Prove that G is abelian and that every non-identity element of G has order 2.
   (d) Fix i, with $1 \leq i \leq n$. Show that there exists $\sigma \in G$ such that $\sigma(\sqrt{p_i}) = -\sqrt{p_i}$ and such that $\sigma(\sqrt{p_j}) = \sqrt{p_j}$ for all $j \neq i$.
   (e) Use the preceding to show that $F = \mathbb{Q}(\sqrt{p_1} + \cdots + \sqrt{p_n})$.