Comprehensive Examination in Algebra
Department of Mathematics, Temple University

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PART I: Do three of the following problems.

1. Let G be a finite group, and let $H \leq G$ be a subgroup of G such that the index $h = |G : H|$ satisfies $h! < |G|$. Show that H contains a normal subgroup of G other than the trivial subgroup containing only the identity element of G.

2. Let F be a field, let V be a finite-dimensional vector space over F, and let $f \colon V \to V$ be an F-linear operator on V (i.e., a linear transformation from V to itself). Prove that the following assertions are equivalent:

   (i) $\operatorname{Ker} f = \operatorname{Ker} f^2$

   (ii) $\operatorname{Im} f = \operatorname{Im} f^2$

   (iii) $V = \operatorname{Im} f \oplus \operatorname{Ker} f$

   Here, $f^2 = f \circ f$, and Ker and Im denote the kernel and the image, respectively, of the linear map in question.

3. Put $R = \{f(x) \in F[x] \mid f'(0) = 0\}$, where F is a field and $f'(x)$ denotes the derivative of $f(x)$; so R consists of all polynomials in $F[x]$ whose coefficient at x equals 0.

   (a) Show that R is a subring of $F[x]$ and that the units of R are the nonzero constant polynomials.

   (b) Show that both x^2 and x^3 are irreducible elements of R.

   (c) Show that R is not a UFD.

4. Let $n \in \mathbb{Z}$ be such that $\sqrt{n} \notin \mathbb{Z}$. Consider the ring $R = \mathbb{Z}[\sqrt{n}] = \{a + b\sqrt{n} \mid a, b \in \mathbb{Z}\}$; this is a subring of $\mathbb{C}$. (You need not prove this.) Prove:

   (a) For every $0 \neq z \in R$ there is a $0 \neq z' \in R$ such that $zz' \in \mathbb{Z}$.

   (b) Conclude from (a) that the quotient ring R/I is finite for every nonzero ideal I of R.

   (c) Conclude from (b) that every nonzero prime ideal of R is maximal.

Part II: Do two of the following problems.

1. Let $f\colon G \twoheadrightarrow H$ be a surjective homomorphism of groups. Prove:

   (a) If P is a Sylow p-subgroup of G then $f(P)$ is a Sylow p-subgroup of H.

   (b) If Q is a Sylow p-subgroup of H then $Q = f(P)$ for some Sylow p-subgroup P of G.

2. Let $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$, for some integer $n > 1$.

   (a) Let $I_1, I_2, \ldots, I_t$ be (finitely many) maximal ideals of R. Prove that $I_1 \cap I_2 \cap \cdots \cap I_t \neq (0)$, where (0) denotes the zero ideal of R.

   (b) Prove that there exists an infinite set $\{I_\alpha \mid \alpha \in T\}$ of maximal ideals of R such that
   $$\bigcap_{\alpha \in T} I_\alpha \neq (0).$$

   (c) Prove that there exists an infinite set $\{I_\beta \mid \beta \in U\}$ of maximal ideals of R such that
   $$\bigcap_{\beta \in U} I_\beta = (0).$$

   (d) A nonzero left R-module M is *simple* provided the only left R-submodules of M are M itself and the zero module. Use (c) to prove that for some set $\{M_\gamma \mid \gamma \in V\}$ of simple left R-modules there exists an injective left R-module homomorphism:
   $$R \to \prod_{\gamma \in V} M_\gamma$$

3. Let $\xi \in \mathbb{C}$ be a primitive 8th root of unity, and let $\sqrt[8]{2}$ be a real 8th root of 2. Let $f(x) = x^8 - 2 \in \mathbb{Q}[x]$, let K be the splitting field in $\mathbb{C}$ of $f(x)$ over $\mathbb{Q}$, and let G be the Galois group of K over $\mathbb{Q}$.

   (a) Prove that $\mathbb{Q}(\xi) = \mathbb{Q}(i, \sqrt{2})$.

   (b) Prove that $K = \mathbb{Q}(i, \sqrt[8]{2})$, and determine the index $[K : \mathbb{Q}]$.

   (c) Prove that G is generated by elements σ and τ such that $\sigma^8 = \tau^2 = 1$ and $\sigma\tau = \tau\sigma^3$.