PART I: Do three of the following problems.

1. Let $p$ be a prime number and let $G$ be a nonabelian group of order $p^3$. Recall that the center $Z(G)$ of $G$ is the normal subgroup of $G$ consisting of all elements $z \in G$ such that $zg = gz$ for all $g \in G$.

   (a) Prove that $|Z(G)| = p$.

   (b) Prove that $G/Z(G) \cong Z_p \times Z_p$, where $Z_p$ is the group of order $p$.

   (You do not have to prove that $Z(G)$ is a normal subgroup of $G$. You may also use other standard facts from group theory without proof, but please quote them.)

2. Let $R$ be a commutative ring with identity, and let $M_2(R)$ be the ring of $2 \times 2$ matrices with entries in $R$. (You do not have to prove that $M_2(R)$ is a ring.)

   (a) Prove that every two-sided ideal of $M_2(R)$ has the form

   $$M_2(I) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in I \right\}$$

   for some ideal $I$ of $R$.

   (b) Say that $M_2(R)$ is prime if for all nonzero two-sided ideals $A$ and $B$ of $M_2(R)$, the ideal product, $AB = \{ \sum_{i=1}^n L_i M_i \mid i = 1, 2, \ldots, L_i \in A, M_i \in B \}$, contains at least one nonzero element. Prove that $M_2(R)$ is prime if and only if $R$ is an integral domain.

3. Let $GL_3(\mathbb{C})$ denote the group of all invertible $3 \times 3$ complex matrices and let $S = \{ A \in GL_3(\mathbb{C}) \mid A^3 = I \}$, where $I$ is the $3 \times 3$ identity matrix.

   (a) Show that if $A \in S$, then all conjugates $C^{-1}AC$ with $C \in GL_3(\mathbb{C})$ also lie in $S$. Conclude that $S$ decomposes into a disjoint union of distinct conjugacy classes in $GL_3(\mathbb{C})$.

   (b) Find the number of conjugacy classes in $S$. Exhibit a representative of each of the conjugacy classes.
4. Let $p$ be a prime number, and let $f(x) = x^p - x + 1 \in \mathbb{F}_p[x]$.

(a) Suppose that $\alpha$ is a root of $f(x)$ in some extension field $K$ of $\mathbb{F}_p$. Prove that $\alpha + 1$ is also a root in $K$ of $f(x)$.
(b) Prove that $f(x)$ is irreducible and separable in $\mathbb{F}_p[x]$.

Part II: Do two of the following problems.

1. Let $G$ be a group having a finite normal subgroup $N$ such that $G/N$ is finitely generated abelian.

(a) Prove that every subgroup of $G$ is finitely generated.
(b) Let $C$ denote the centralizer of $N$ in $G$ (i.e., the set of elements $z \in G$ such that $zn = nz$ for all $n \in N$). Prove that $C$ is a normal subgroup of $G$ of finite index.
(c) For $a, c \in C$, put $f_a(c) = acc^{-1}c^{-1}$. Show that $f_a(c) \in C \cap N$ and that $aca^{-1} = f_a(c)c$. Use this to show that $f_a$ is a group homomorphism from $C$ to $C \cap N$.
(d) Let $S$ be a (finite) generating set for $C$. Prove that the intersection $\bigcap_{a \in S} \text{Ker} f_a$ is exactly the center $Z(C)$ of $C$.
(e) Conclude that $A = Z(C)$ is a finitely generated abelian normal subgroup of $G$ such that $G/A$ is finite.

2. Let $F$ be a field, and set $R = F[x, y, z]/(z^2 - xy)$.

(a) Prove that $z^2 - xy$ is an irreducible element of $F[x, y, z]$.
(b) Prove that $R$ is an integral domain.
(c) Prove that $R$ is not a UFD.

3. Let $E/K$ be a finite Galois extension of fields, with Galois group $G = \text{Gal}(E/K)$. Let $K \subseteq F_i \subseteq E$ ($i = 1, 2$) be two intermediate fields and let $F_1F_2$ denote the subfield of $E$ that is generated by $F_1$ and $F_2$.

(a) Show that $E$ is Galois over each $F_i$.
(b) Put $G_i = \text{Gal}(E/F_i)$; so each $G_i$ is a subgroup of $G$. (You don’t need to prove this.) Show that $E = F_1F_2$ if and only if $G_1 \cap G_2 = 1$ and that $G$ is generated by $G_1$ and $G_2$ if and only if $F_1 \cap F_2 = K$.
(c) Show that $G$ is the direct product of $G_1$ and $G_2$ if and only if the following three conditions hold: both $F_i$ are Galois over $K$, $F_1 \cap F_2 = K$, and $E = F_1F_2$. 

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