Comprehensive Examination in Algebra
Department of Mathematics, Temple University

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PART I: Do three of the following problems.

1. Let p be a prime number and let G be a nonabelian group of order p^3. Recall that the *center* $Z(G)$ of G is the normal subgroup of G consisting of all elements $z \in G$ such that $zg = gz$ for all $g \in G$.

   (a) Prove that $|Z(G)| = p$.

   (b) Prove that $G/Z(G) \cong Z_p \times Z_p$, where Z_p is the group of order p.

   (You do not have to prove that $Z(G)$ is a normal subgroup of G. You may also use other standard facts from group theory without proof, but please quote them.)

2. Let R be a commutative ring with identity, and let $M_2(R)$ be the ring of 2×2 matrices with entries in R. (You do not have to prove that $M_2(R)$ is a ring.)

   (a) Prove that every two-sided ideal of $M_2(R)$ has the form

   $$M_2(I) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in I \right\}$$

   for some ideal I of R.

   (b) Say that $M_2(R)$ is *prime* if for all nonzero two-sided ideals A and B of $M_2(R)$, the *ideal product*, $AB = \{\sum_{i=1}^{n} L_i M_i \mid i = 1, 2, \ldots, \; L_i \in A, \; M_i \in B\}$, contains at least one nonzero element. Prove that $M_2(R)$ is prime if and only if R is an integral domain.

3. Let $\mathrm{GL}_3(\mathbb{C})$ denote the group of all invertible 3×3 complex matrices and let $S = \{A \in \mathrm{GL}_3(\mathbb{C}) \mid A^3 = I\}$, where I is the 3×3 identity matrix.

   (a) Show that if $A \in S$, then all conjugates $C^{-1}AC$ with $C \in \mathrm{GL}_3(\mathbb{C})$ also lie in S. Conclude that S decomposes into a disjoint union of distinct conjugacy classes in $\mathrm{GL}_3(\mathbb{C})$.

   (b) Find the number of conjugacy classes in S. Exhibit a representative of each of the conjugacy classes.

4. Let p be a prime number, and let $f(x) = x^p - x + 1 \in \mathbb{F}_p[x]$.

   (a) Suppose that α is a root of $f(x)$ in some extension field K of $\mathbb{F}_p$. Prove that $\alpha+1$ is also a root in K of $f(x)$.

   (b) Prove that $f(x)$ is irreducible and separable in $\mathbb{F}_p[x]$.

Part II: Do two of the following problems.

1. Let G be a group having a finite normal subgroup N such that G/N is finitely generated abelian.

   (a) Prove that every subgroup of G is finitely generated.

   (b) Let C denote the centralizer of N in G (i.e., the set of elements $z \in G$ such that $zn = nz$ for all $n \in N$). Prove that C is a normal subgroup of G of finite index.

   (c) For $a, c \in C$, put $f_a(c) = aca^{-1}c^{-1}$. Show that $f_a(c) \in C \cap N$ and that $aca^{-1} = f_a(c)c$. Use this to show that f_a is a group homomorphism from C to $C \cap N$.

   (d) Let S be a (finite) generating set for C. Prove that the intersection $\bigcap_{a \in S} \operatorname{Ker} f_a$ is exactly the center $Z(C)$ of C.

   (e) Conclude that $A = Z(C)$ is a finitely generated abelian normal subgroup of G such that G/A is finite.

2. Let F be a field, and set $R = F[x, y, z]/(z^2 - xy)$.

   (a) Prove that $z^2 - xy$ is an irreducible element of $F[x, y, z]$.

   (b) Prove that R is an integral domain.

   (c) Prove that R is not a UFD.

3. Let E/K be a finite Galois extension of fields, with Galois group $G = \operatorname{Gal}(E/K)$. Let $K \subseteq F_i \subseteq E$ $(i = 1, 2)$ be two intermediate fields and let F_1F_2 denote the subfield of E that is generated by F_1 and F_2.

   (a) Show that E is Galois over each F_i.

   (b) Put $G_i = \operatorname{Gal}(E/F_i)$; so each G_i is a subgroup of G. (You don't need to prove this.) Show that $E = F_1F_2$ if and only if $G_1 \cap G_2 = 1$ and that G is generated by G_1 and G_2 if and only if $F_1 \cap F_2 = K$.

   (c) Show that G is the direct product of G_1 and G_2 if and only if the following three conditions hold: both F_i are Galois over K, $F_1 \cap F_2 = K$, and $E = F_1F_2$.