Part I: Do three of the following problems

1. Let $R = \mathbb{Z}[x]$, and let $I$ be a nonzero ideal of $R$.
   (i) Let $J = \{ a \in \mathbb{Z} : a = 0 \text{ or } a \text{ is the leading coefficient of a polynomial in } I \}$. Prove that $J$ is an ideal of $\mathbb{Z}$.
   (ii) Recall, if $t$ is a positive integer, that $I^t$ denotes the ideal of $R$ generated by $\{ f^t : f \in I \}$. Prove that $I^t = I^{t+1}$ if and only if $I = R$.

2. Let $n$ be a positive integer, and let $X$ and $Y$ be invertible $n \times n$ complex matrices such that $X^{-1} Y X = e^{2\pi i/n} Y$. Determine the Jordan Form of $Y$.

3. Let $\mathbb{Q}^+$ denote the additive group of rational numbers, and let $\mathbb{Z}^+$ denote the additive group of integers. Prove that $\mathbb{Q}^+/\mathbb{Z}^+$ is not finitely generated.

4. Let $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Determine $[K : \mathbb{Q}]$. 
Part II: Do two of the following problems

1. Let $G = \text{GL}_2(\mathbb{F}_p)$ be the group of invertible $2 \times 2$-matrices over the field $\mathbb{F}_p$ with $p$ elements ($p$ a prime). Determine the number of Sylow $p$-subgroups of $G$.

2. Let $R$ be a ring with multiplicative identity $1$. Recall that a nonzero left ideal $L$ of $R$ is said to be minimal if $L$ is simple as a left $R$-module.

   (i) Let $M$ be a simple left $R$-module, and let $I$ be the sum of all of the minimal left ideals of $R$ isomorphic as left $R$-modules to $M$. (In other words, $I$ is generated by the union of all of the minimal left ideals isomorphic to $M$.) Prove that $I$ is a two-sided ideal of $R$.

   (ii) Let $J$ denote the sum of all of the minimal left ideals of $R$, and assume that $J = R$. Prove that $R$ is then the sum of some finite collection of minimal left ideals. (Hint: This conclusion does not hold true for rings without multiplicative identities.)

   (iii) Prove that if $C$ is a commutative integral domain containing a minimal left ideal then $C$ is a field.

3. Let $F/K$ be a finite Galois extension of fields and let $G = \text{Gal}(F/K)$ be its Galois group. Furthermore, let $E/K$ be a non-trivial subextension; so $K \subseteq E \subseteq F$ and $E \neq K$.

   (i) Assume that $G$ is nilpotent. Show that $E/K$ contains a non-trivial Galois extension $E'/K$.

   (ii) Assume that $G$ is solvable and that $E/K$ is Galois. Show that $E/K$ contains a non-trivial Galois extension $E'/K$ such that $\text{Gal}(E'/K)$ is abelian.