PH.D. COMPREHENSIVE EXAMINATION
ABSTRACT ALGEBRA SECTION

Fall 1995

Part I. Do three (3) of these problems.

I.1. Show that the alternating group A_6 has no subgroup of prime index.

I.2. Let R be a commutative domain in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that R is a field of positive characteristic. Is R necessarily finite?

I.3. Let V be a finite-dimensional vector space and let $\phi : V \to V$ be an endomorphism. Suppose that, for some $v \in V$ and $k \geq 1$, $\phi^k(v) = 0$ but $\phi^{k-1}(v) \neq 0$. Prove:

(1) The subspace W of V that is generated by $\{v, \phi(v), \ldots, \phi^{k-1}(v)\}$ is ϕ-invariant (i.e., $\phi(W) \subseteq W$) and satisfies $\dim(W) = k$.
(2) The minimal polynomial $m(X)$ of ϕ is divisible by X^k.

I.4. Let F be a field of characteristic $p > 0$ and let $f(X) \in F[X]$ be an irreducible polynomial which is not separable (i.e., $f(X)$ has repeated roots). Show that $f(X) = g(X^p)$ for some irreducible polynomial $g(X) \in F[X]$.

Part II. Do two (2) of these problems.

II.1. Show that there is no simple group of order 56 (without quoting Burnside's $p^a q^b$-Theorem or special cases thereof).

II.2. Let $M \neq 0$ be a finitely generated torsion module over a commutative PID R.

(1) Show that M is *indecomposable* (i.e., M is not the direct sum of two nonzero submodules) if and only if $M \cong R/p^n R$ for some irreducible element p of R and some $n > 0$.
(2) Show that M is *irreducible* (i.e., M has no submodules other than 0 and M) if and only if $M \cong R/pR$ for some irreducible element p of R.

II.3. Let F be a finite field and n a positive integer. Prove that there exists an irreducible polynomial over F of degree n.