PhD Algebra Exam
Fall 1987

Part I: Do three of these problems.

1. Let G be a finite group.

a) Prove that if $\phi(x) = x^2$ is a homomorphism $G \to G$ then G is abelian.
b) Give an example of a finite non-abelian group G such that $\psi(x) = x^4$ is a homomorphism.

2. Let A be a 3×3 matrix with rational entries. Recall that A is called nilpotent if for some positive integer k, $A^k = 0$.

a) Show that A is nilpotent $\iff$ 0 is the unique eigenvalue of A.

Now let A be nilpotent.

b) Show that $A^3 = 0$.
c) Exhibit all possible Jordan forms of A.

3. Let $R = M_2(\mathbb{Q})$, the ring of 2×2 matrices over $\mathbb{Q}$.

a) Prove that R is simple; that is, R has no proper non-zero ideals.
b) Exhibit a proper non-zero right ideal of R.

4. Let $a = 1 + \sqrt{3}$, let $\alpha = \sqrt{a}$, and let $K = \mathbb{Q}(\alpha)$.

a) Find the irreducible polynomial for α over $\mathbb{Q}$.
b) Let $F = \mathbb{Q}(a)$. Show that F is normal over $\mathbb{Q}$, and K is normal over F.
c) Show that K is not normal over $\mathbb{Q}$.

Part II: Do two of these problems.

5. Prove that all groups of order ≤ 12 are solvable.

6. Recall that if R is a ring with 1, a unit of R is an element with a multiplicative inverse in R. Consider $R = \mathbb{Z}[\sqrt{q}]$, for $q \in \mathbb{Z}$. Define $N(a + b\sqrt{q}) = a^2 - qb^2$.

a) Show that $u \in R$ is a unit $\iff$ $Nu = \pm 1$.
b) Find all the units of $\mathbb{Z}[\sqrt{-3}]$.
c) Show that $\mathbb{Z}[\sqrt{2}]$ has infinitely many units.

7. Let $F = \mathbb{Q}(\sqrt[4]{5})$.

a) Find the degree and a basis of F over $\mathbb{Q}$.
b) Find the group of automorphisms of F over $\mathbb{Q}$ and describe its fixed field.
c) Describe the Galois closure of F over $\mathbb{Q}$ and its Galois group.