PhD Algebra Exam
Fall 1987

Part I: Do three of these problems.

1. Let $G$ be a finite group.
   a) Prove that if $\phi(x) = x^2$ is a homomorphism $G \to G$ then $G$ is abelian.
   b) Give an example of a finite non-abelian group $G$ such that $\psi(x) = x^4$ is a homomorphism.

2. Let $A$ be a $3 \times 3$ matrix with rational entries. Recall that $A$ is called nilpotent if for some positive integer $k$, $A^k = 0$.
   a) Show that $A$ is nilpotent $\iff$ 0 is the unique eigenvalue of $A$.
   Now let $A$ be nilpotent.
   b) Show that $A^3 = 0$.
   c) Exhibit all possible Jordan forms of $A$.

3. Let $R = M_2(\mathbb{Q})$, the ring of $2 \times 2$ matrices over $\mathbb{Q}$.
   a) Prove that $R$ is simple; that is, $R$ has no proper non-zero ideals.
   b) Exhibit a proper non-zero right ideal of $R$.

4. Let $a = 1 + \sqrt{3}$, let $\alpha = \sqrt{a}$, and let $K = \mathbb{Q}(\alpha)$.
   a) Find the irreducible polynomial for $\alpha$ over $\mathbb{Q}$.
   b) Let $F = \mathbb{Q}(a)$. Show that $F$ is normal over $\mathbb{Q}$, and $K$ is normal over $F$.
   c) Show that $K$ is not normal over $\mathbb{Q}$.

Part II: Do two of these problems.

5. Prove that all groups of order $\leq 12$ are solvable.

6. Recall that if $R$ is a ring with 1, a unit of $R$ is an element with a multiplicative inverse in $R$. Consider $R = \mathbb{Z}[\sqrt{q}]$, for $q \in \mathbb{Z}$. Define $N(a + b\sqrt{q}) = a^2 - qb^2$.
   a) Show that $u \in R$ is a unit $\iff Nu = \pm 1$.
   b) Find all the units of $\mathbb{Z}[\sqrt{-3}]$.
   c) Show that $\mathbb{Z}[\sqrt{2}]$ has infinitely many units.

7. Let $F = \mathbb{Q}(\sqrt{5})$.
   a) Find the degree and a basis of $F$ over $\mathbb{Q}$.
   b) Find the group of automorphisms of $F$ over $\mathbb{Q}$ and describe its fixed field.
   c) Describe the Galois closure of $F$ over $\mathbb{Q}$ and its Galois group.