On optimal short-term recurrences for generating orthogonal Krylov subspace bases

ABSTRACT: Any Krylov subspace method in some way needs to generate bases of the underlying Krylov subspaces. For stability reasons the generated bases should be well-conditioned (at best orthogonal), and for efficiency reasons the bases should be generated by a short-term recurrence. In this talk we will discuss when (in exact arithmetic) these two goals can be achieved simultaneously. More precisely, we will analyze necessary and sufficient conditions on a nonsingular matrix $A$, such that for any initial vector $r_0$, an orthogonal basis of the Krylov subspaces $K_n(A, r_0)$ is generated by an optimal short-term recurrence. Orthogonality here is meant with respect to a given (but unspecified) positive definite inner product. The conditions on $A$ we will deal with were first derived and characterized more than 20 years ago in the fundamental paper of Faber and Manteuffel (SINUM 21, No. 2, pp. 352-362). Their main theorem is often quoted and appears to be widely known. Its details and underlying concepts, however, are quite intricate, with some subtleties not covered in the literature we are aware of. The talk will have the following two main purposes: First, to present and clarify some of the known results in the context of the Faber-Manteuffel Theorem. Second, to review attempts to find an easier proof of the theorem, and to explain what remains to be done to complete work in this direction.

The talk will be based on joint work with Paul Saylor and Zdeněk Strakoš.