Co-Poisson coalgebras and (co-)Poisson Hopf algebras

Q. -S. Wu

School of Mathematical Sciences, Fudan University
Joint with Qi LOU

Talk at the Conference in honor of Ellen and Martin
Temple University, Philadelphia, July 27, 2017
Outline

1 Motivation

2 Poisson structures and copoison structures
   - Poisson algebra
   - co-Poisson coalgebra
   - Poisson Hopf algebra
   - co-Poisson Hopf algebra

3 Duality between Poisson and co-Poisson structures

4 (Co-) Poisson structures on \( k[x_1, \ldots, x_d] \)
   - Co-Poisson coalgebra structure on \( A \)
   - Poisson Hopf algebra structure on \( A \)
   - Co-Poisson Hopf algebra structure on \( A \)
Our Motivation

- Poisson structure (or algebra) is a very active subject of research in mathematics (and mathematical physics) such as: differential geometry, Lie groups, quantum groups, non-commutative geometry, non-commutative algebra and representation theory.

- Co-Poisson structure (or coalgebra) is a dual concept of Poisson structure in categorial point of view. It arises also in mathematics and mathematical physics naturally.
• The category of connected and simply-connected Lie groups is equivalent to the category of finite-dimensional Lie algebras.

• $\mathcal{O}(G) \cong U(\mathfrak{g})^\circ$, where $\mathfrak{g}$ is the corresponding Lie algebra of Lie group $G$, $U(\mathfrak{g})^\circ$ is the Hopf dual of the enveloping algebra $U(\mathfrak{g})$.

• A Lie group $G$ is a **Poisson Lie group** if and only if $\mathcal{O}(G)$ is a Poisson Hopf algebra.

• The category of connected and simply-connected Poisson Lie groups is equivalent to the category of finite-dimensional Lie bialgebras.
• The Lie bialgebra structures on any Lie algebra $\mathfrak{g}$ is in one-to-one correspondence with the co-Poisson Hopf structures on $U(\mathfrak{g})$.

• In this case, $\mathcal{O}(G) \cong U(\mathfrak{g})^\circ$ as Poisson Hopf algebra, where $\mathfrak{g}$ is the corresponding Lie bialgebra of Poisson Lie group $G$.

• To quantize a Lie group or Lie algebra one should equip it with an extra structure, namely, a Poisson Lie group structure or Lie bialgebra structure, respectively.

• Therefore co-Poisson structure naturally appears in the theory of quantum groups and in mathematical physics.
Let $t_n : V^\otimes n \to V^\otimes n$ be the map

$v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}$.

Suppose $(A, \mu, \eta)$ is an algebra. $[-, -] = \mu - \mu \circ t_2$, i.e.,

$[a, b] = ab - ba$ is the commutator.

Suppose $(C, \Delta, \varepsilon)$ is a coalgebra.

$\Delta(c) = \sum c_1 \otimes c_2$ and $(\Delta \otimes 1)\Delta(c) = \sum c_1 \otimes c_2 \otimes c_3$.

Let $\Delta^{(2)} = (\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta : C \to C \otimes C \otimes C$, and

$\Delta' = \Delta - t_2 \circ \Delta$ be the cocommutator.
Definition 2.1

An algebra $A$ equipped with a linear map $\{-, -\} : A \otimes A \to A$ is called a Poisson algebra if

1. $A$ with $\{-, -\} : A \otimes A \to A$ is a Lie algebra;
2. $\{a, -\} : A \to A$ is a derivation with respect to the multiplication of $A$ for all $a \in A$, that is, $\{a, bc\} = \{a, b\}c + b\{a, c\}$ for all $b, c \in A$. 
Motivation
Poisson structures and copoison structures
Duality between Poisson and co-Poisson structures
(Co-) Poisson structures on $k[x_1, \cdots, x_d]$

Poisson algebra
co-Poisson coalgebra
Poisson Hopf algebra
co-Poisson Hopf algebra

Poisson algebra

Definition 2.2

An algebra $(A, \mu, \eta)$ equipped with a linear map $p : A \otimes A \to A$ is called a **Poisson algebra** if

1. $A$ with $p : A \otimes A \to A$ is a Lie algebra, i.e.,

   $p \circ (1 + t_2) = 0, \quad \text{(skew-symmetric)}$

   $p \circ (p \otimes 1) \circ (1 + t_3 + t_3^2) = 0; \quad \text{(Jacobi identity)}$

2. $p(1 \otimes \mu) = \mu(p \otimes 1) - \mu(1 \otimes p)t^2_3. \quad \text{(Leibnitz rule)}$

Remark 2.3

We don’t assume that $A$ is commutative here.
The following is a result of Farkas and Letzter [FL, Theorem 1.2].
Proposition 2.4

Suppose that \( A \) is a prime Poisson algebra with \( \{ A, A \} \neq 0 \) and \( [A, A] \neq 0 \). Then, for any \( a, b \in A \),

1. the following map is a bimodule isomorphism
   \[
   f_{a,b} : A[a, b]A \rightarrow A\{a, b\}A, \quad x[a, b]y \mapsto x\{a, b\}y.
   \]

2. In the Martindale ring of quotients of \( A \), \( \{x, y\} = \alpha[x, y] \) where \( \alpha \) is the element represented by any \( f_{a,b} \neq 0 \).

It follows that there is no nontrivial Poisson algebra structure on any simple algebras such as \( A_n(k) \) and \( M_n(k) \).
co-Poisson coalgebra

Definition 2.5

A coalgebra \((C, \Delta, \varepsilon)\) equipped with a linear map \(q : C \rightarrow C \otimes C\) is called a co-Poisson coalgebra if

1. \(C\) with \(q : C \rightarrow C \otimes C\) is a Lie coalgebra, i.e.,
   \[
   (1 + t_2) \circ q = 0, \quad \textit{(skew-symmetric)}
   \]
   \[
   (1 + t_3 + t_3^2) \circ (q \otimes 1) \circ q = 0; \quad \textit{(co-Jacobi identity)}
   \]

2. \((1 \otimes \Delta)q = (q \otimes 1)\Delta - t_3(1 \otimes q)\Delta. \quad \textit{(co-Leibnitz rule)}\)

The cocommutator \(\Delta’\) gives a co-Poisson coalgebra structure on any coalgebra \((C, \Delta, \varepsilon)\).
In a co-Poisson coalgebra \((C, q)\), we use the sigma notation

\[ q(c) = \sum c(1) \otimes c(2) \quad \text{and}, \]

\[ (q \otimes 1)q(c) = \sum c(1) \otimes c(2) \otimes c(3), \]

where \(\sum\) is also often omitted in the computations.

Then,

\[ (1 \otimes q)q(c) = (1 \otimes q)(-c(2) \otimes c(1)) = -c(3) \otimes c(1) \otimes c(2) = c(3) \otimes c(2) \otimes c(1). \]
Remark 2.6

*By using the sigma notation, the co-Leibnitz rule reads as*

\[ c_{(1)} \otimes c_{(2)}^1 \otimes c_{(2)}^2 = c_{1(1)} \otimes c_{1(2)} \otimes c_2 - c_{2(2)} \otimes c_1 \otimes c_{2(1)}, \]

*for all* \( c \in C \).

*It is equivalent to*

\[ c_{(1)}^1 \otimes c_{(1)}^2 \otimes c_{(2)} = c_1 \otimes c_{2(1)} \otimes c_{2(2)} - c_{1(2)} \otimes c_2 \otimes c_{1(1)}, \]

*i.e.,*

\[ (\Delta \otimes 1)q = (1 \otimes q)\Delta - t_3^2(q \otimes 1)\Delta \quad (\text{co-Leibnitz rule}) \]
Remark 2.7

If $C$ is cocommutative, then the co-Leibnitz rule is equivalent to

$$\left(\Delta \otimes 1\right)q = (1 - t_3)(1 \otimes q)\Delta \quad (co-Leibnitz \ rule)$$

There is no non-trivial co-Poisson coalgebra structure on any group algebra $k[G]$ (By checking the co-Leibniz rule.)
Dual to $\{a, 1\} = 0$ (because $\{a, -\}$ is a derivation), that is, $p(1 \otimes \eta) = 0$ in a Poisson algebra.

Proposition 2.8

Let $(C, \Delta, \varepsilon, q)$ be a co-Poisson coalgebra. Then

$$(\varepsilon \otimes 1) \circ q = (1 \otimes \varepsilon) \circ q = 0,$$

i.e.,

$$\varepsilon(h_{(1)})h_{(2)} = h_{(1)}\varepsilon(h_{(2)}) = 0.$$
Poisson Hopf algebra

Definition 2.9

A Hopf algebra \((H, \mu, \eta; \Delta, \varepsilon; S)\) with a linear map \(\{-, -\} : H \otimes H \to H\) is called a **Poisson Hopf algebra** if

1. \((H, \{-, -\})\) is a Poisson algebra;
2. The structures are compatible: for all \(a, b \in H\),

\[
\Delta (\{a, b\}) = \sum \{a_1, b_1\} \otimes a_2 b_2 + \sum a_1 b_1 \otimes \{a_2, b_2\} \quad (2.1)
\]
Let $A$ and $B$ be two Poisson algebras. An algebra morphism $f : A \rightarrow B$ is a **Poisson algebra morphism** if $fp_A = p_B(f \otimes f)$, that is, $f(\{a, b\}_A) = \{f(a), f(b)\}_B$ for all $a, b \in A$.

**Remark 2.10**

Let $A$ and $B$ be two commutative Poisson algebra. Then there is a Poisson structure on $A \otimes B$ given by

$$\{a \otimes b, a' \otimes b'\} = \{a, a'\} \otimes bb' + aa' \otimes \{b, b'\}$$

for all $a, a' \in A$ and $b, b' \in B$.

If $H$ is commutative, then the compatible condition (2.1) in Definition 2.9 means that $\Delta : H \rightarrow H \otimes H$ is a Poisson algebra morphism.
Proposition 2.11 (L. I. Korogodski, Y. S. Soibelman 1998)

Let $H$ be a Poisson Hopf algebra. Then,

1. the counit $\varepsilon : H \to k$ is a Poisson algebra morphism.

2. If $H$ is commutative, then $\Delta : H \to H \otimes H$ is a Poisson algebra morphism.

3. If $H$ is commutative, then the antipode $S : H \to H$ is a Poisson algebra anti-morphism.
Proposition 2.12

Let $\mathfrak{g}$ be a non-abelian Lie algebra over a field of characteristic $\neq 2$. Then there is no nontrivial Poisson Hopf structure on $U(\mathfrak{g})$.

Proposition 2.13

There is no nontrivial Poisson Hopf structure on any group algebra $k(G)$. 
Definition 2.14

A Hopf algebra \((H, \mu, \eta; \Delta, \varepsilon; S)\) equipped with a linear map \(q : H \to H \otimes H\) is called a **co-Poisson Hopf algebra** if

1. \(H\) with \(q : H \to H \otimes H\) is a co-Poisson coalgebra.
2. \(q\) is a \(\Delta\)-derivation, i.e., for all \(a, b \in H\),

\[
q(ab) = q(a)\Delta(b) + \Delta(a)q(b) \quad (2.2)
\]
Let $C$ and $D$ be two co-Poisson coalgebras. A coalgebra morphism $g : C \to D$ is called a **co-Poisson coalgebra morphism** if

$$(g \otimes g)q_C = q_D g.$$  

**Remark 2.15**

Let $C$ and $D$ be two cocommutative co-Poisson coalgebras. Then $C \otimes D$ has a co-Poisson structure $q_{C \otimes D}$ being defined as

$$C \otimes D \xrightarrow{(1 \otimes \tau \otimes 1)(q_C \otimes \Delta_D + \Delta_C \otimes q_D)} C \otimes D \otimes C \otimes D,$$

that is,

$$q_{C \otimes D}(c \otimes d) = c_{(1)} \otimes d_1 \otimes c_{(2)} \otimes d_2 + c_1 \otimes d_{(1)} \otimes c_2 \otimes d_{(2)}.$$  

Hence (2.2) in Definition 2.14 is equivalent to say $\mu : H \otimes H \to H$ is a co-Poisson coalgebra morphism.
Proposition 2.16

Let \((H, \mu, \eta; \Delta, \varepsilon; S, q)\) be a co-Poisson Hopf algebra. Then

1. The unit \(\eta\) is a co-Poisson coalgebra morphism.
2. If \(H\) is cocommutative, then \(\mu : H \otimes H \rightarrow H\) is a co-Poisson coalgebra morphism.
3. If \(H\) is cocommutative, then \(S\) is a co-Poisson coalgebra anti-morphism.
Example 2.17

Let $H_4 = k\langle 1, g, x, gx \mid g^2 = 1, x^2 = 0, xg = -gx \rangle$ be the 4-dimensional Sweedler’s Hopf algebra with char $k \neq 2$.

1. Every Poisson algebra structure on $H_4$ is given by $
   \{g, x\} = \lambda x + \mu gx$ for some $\lambda, \mu \in k$.

2. There is no nontrivial Poisson Hopf algebra structure on $H_4$.

3. Every co-Poisson structure on $H_4$ is given by $q(1) = q(g) = 0,
   q(x) = \alpha \Delta'(x), q(gx) = \beta \Delta'(gx)$ for some $\alpha, \beta \in k$.

4. There is no nontrivial co-Poisson Hopf algebra structure on $H_4$. 
Proposition 3.1

Let $C$ be a coalgebra, $q : C \to C \otimes C$ be a linear map. Then, 
$(C, q)$ is a co-Poisson coalgebra $\iff$ $C^*$ with $q^* : C^* \otimes C^* \to C^*$ is a Poisson algebra.

If $g : C \to D$ is a co-Poisson coalgebra morphism, then $g^* : D^* \to C^*$ is a Poisson algebra morphism.
Recall for an algebra $A$,
$A^\circ = \{ f \in A^* \mid \ker f \text{ contains a cofinite (left/right) ideal } I \text{ of } A \}$.

**Proposition 3.2**

Let $A$ be a Poisson algebra. If $A$ is a left or right noetherian, then $A^\circ$ is a co-Poisson coalgebra.

**Example 3.3**

Let $A = k[x_1, x_2, \cdots, x_n, \cdots]$ be a polynomial algebra with variables $\{x_i \mid i \geq 1\}$. Let $p(x_i \otimes x_j) = \{x_i, x_j\} = 1$ for all $i < j$. Then $p$ gives a Poisson algebra structure on $A$. Let $\varepsilon : A \to k, x_i \mapsto 0$ be the augmentation map. Then $\varepsilon \in A^\circ$, but $p^*(\varepsilon) = \varepsilon p \notin A^\circ \otimes A^\circ \cong (A \otimes A)^\circ$. 
Theorem 3.4 (L. I. Korogodski, Y. S. Soibelman 1998)

Let $H$ be a left or right noetherian Poisson Hopf algebra. Then $H^\circ$ is a co-Poisson Hopf algebra.

Theorem 3.4 is stated in [KS, Proposition 3.1.5] without noetherian hypothesis. Without this hypothesis, it is not true as showed in the following example.
Motivation
Poisson structures and copoisson structures
Duality between Poisson and co-Poisson structures
(Co-) Poisson structures on $k[x_1, \ldots, x_d]$

Example 3.5

Let $A = k[x_1, x_2, \cdots]$ with the Poisson Hopf algebra structure by letting

$$\{x_1, x_i\} = 0 \text{ for all } i \geq 2,$$

for $1 < i < j \in \mathbb{N}$,

$$\{x_i, x_j\} = \begin{cases} x_1, & j = i + 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\{-, -\}^*(A^\circ) \not\subseteq A^\circ \otimes A^\circ$, thus $\{-, -\}^*$ is not a co-Poisson Hopf structure on $A^\circ$. 

Q. -S. Wu
Co-Poisson coalgebras and (co-)Poisson Hopf algebras
Theorem 3.6 (L. I. Korogodski, Y. S. Soibelman 1998)

Let $H$ be a co-Poisson Hopf algebra. Then the Hopf dual $H^\circ$ is a Poisson Hopf algebra.

Oh and Park prove that the Hopf dual $H^\circ$ of a co-Poisson Hopf algebra $H$ is a Poisson Hopf algebra when $H$ is an almost normalizing extension over $k$, suggested by the $U(g)$ case.

This is true in general. A complete proof is given in [LW] and [Oh] in 2015.
Co-Poisson structures on $k[x_1, \ldots, x_d]$

Let $g = kx_1 \oplus kx_2 \oplus \cdots \oplus kx_d$ be the $d$-dimensional abelian Lie algebra. Then $A = U(g) = k[x_1, \ldots, x_d]$ is a Hopf algebra. Note that $g = P(A)$.  

Let $\mathcal{H}(A)$ be the standard $k$-basis of $A$ which contains all monic monomials.

In the following, for $a \in A$, $\Delta(a) = \sum a_1 \otimes a_2$ is always assumed to be the expression by the standard $k$-basis of $k[x_1, x_2, \cdots, x_d]$.

Denote $\mathcal{I} = \bigoplus_{1 \leq i < j \leq d} k(x_i \otimes x_j - x_j \otimes x_i)$. 
Lemma 4.1

Let $C$ be a coalgebra, $X \in C \otimes C$. Then

$$X \in \mathcal{I} \iff (1 + t_2)X = 0 \text{ and } (\Delta \otimes 1)(X) = (1 - t_3)(1 \otimes X).$$

Lemma 4.2

Let $B$ be a bialgebra. If $X \in B \otimes B$ is skew-symmetric, then so is $X\Delta(x)$ for any $x \in P(B)$. 
Theorem 4.3 (Reciprocity law)

Let $q : A \rightarrow A \otimes A$ and $I : A \rightarrow A \otimes A$ be two linear maps. Then

$$I(a) = (-1)^{|a_2|} q(a_1) \Delta(a_2) \quad \text{for all} \quad a \in A$$

\[ \Updownarrow \]

$$q(a) = I(a_1) \Delta(a_2) \quad \text{for all} \quad a \in A.$$
Theorem 4.4

A linear map \( q : A \to A \otimes A \) gives a co-Poisson coalgebra structure on \( A \) if and only if there is a linear map \( I : A \to A \otimes A \) such that,

1. The image of \( I \) is contained in \( I \).
2. \( q(a) = I(a_1) \Delta(a_2) \) for all \( a \in A \).
3. The co-Jacobi identity holds for \( q \).
Theorem 4.5 (Continued)

In this case, we may assume that, for any \( a \in \mathcal{H}(A) \),

\[
I(a) = \sum_{1 \leq i, j \leq d} \lambda_{a}^{ij} x_i \otimes x_j \in \mathcal{I}
\]

with \( (\lambda_{a}^{ij})_{d \times d} \in M_d(k) \) skew-symmetric. Then the co-Jacobi identity holds for \( q \) if and only if for all \( 1 \leq i < j < k \leq d \) and \( a \in A \),

\[
\sum_{s=1}^{d} \left( \lambda_{a_1}^{sk} \lambda_{x_s a_2}^{ij} + \lambda_{a_1}^{si} \lambda_{x_s a_2}^{jk} + \lambda_{a_1}^{sj} \lambda_{x_s a_2}^{ki} \right) = 0.
\]
Proposition 4.6

Let \( A = k[x, y] \). Then there is an one-to-one correspondence between the co-Poisson structures \( q \) on \( A \) and the linear maps \( l : A \to \mathcal{I} = k(x \otimes y - y \otimes x) \), given by

\[
(l : A \to \mathcal{I} \subseteq A \otimes A) \mapsto (q : A \to A \otimes A, a \mapsto l(a_1)\Delta(a_2)), \quad \text{and}
\]

\[
(q : A \to A \otimes A) \mapsto (l : A \to A \otimes A, a \mapsto (-1)^{|a_2|}q(a_1)\Delta(a_2)).
\]
This is dual to [CAP, Proposition 1.8] in some sense.

**Proposition 4.7**

Any Poisson algebra structure on \( A = k[x_1, \cdots, x_d] \) is given by \( \{x_i, x_j\} = f_{ij} \) where \( \{f_{ij}\}_{d \times d} \) is a skew-symmetric matrix over \( A \) such that for all \( 1 \leq i < j < k \leq d \),

\[
\sum_{l=1}^{d} \left( f_{lk} \frac{\partial f_{ij}}{\partial x_l} + f_{li} \frac{\partial f_{jk}}{\partial x_l} + f_{lj} \frac{\partial f_{ki}}{\partial x_l} \right) = 0. \tag{4.1}
\]
Proposition 4.8

Any Poisson Hopf structure on \( A = k[x_1, \cdots, x_d] \) is given by

\[
\{x_i, x_j\} = \sum_{l=1}^{d} \lambda_{ij}^l x_l \quad (1 \leq i, j \leq d),
\]

where \( \lambda_{ij}^l = -\lambda_{ji}^l \), subject to the relations, for any \( 1 \leq i < j < k \leq d \) and any \( 1 \leq s \leq d \),

\[
\sum_{l=1}^{n} \left( \lambda_{ij}^l \lambda_{lk}^s + \lambda_{jk}^l \lambda_{li}^s + \lambda_{ki}^l \lambda_{lj}^s \right) = 0.
\]
Theorem 4.9

A linear map $q : A \rightarrow A \otimes A$ gives a co-Poisson Hopf structure on $A$ if and only if there exists a linear map $l : A \rightarrow I$, such that for any $a \in A$,

1. $l(a) = 0$ if $a \neq x_i \ (1 \leq i \leq d)$.
2. $q(a) = l(a_1)\Delta(a_2)$.
3. the co-Jacobi identity holds.
Theorem 4.10

Any co-Poisson Hopf structure \( q \) on \( A \) is given by

\[
q(x_s) = \sum_{1 \leq i, j \leq d} \lambda^{ij}_s x_i \otimes x_j \quad (1 \leq s \leq d),
\]

with \( \lambda^{ij}_s = -\lambda^{ji}_s \), subject to the relations, for any \( 0 \leq i < j < k \leq d \) and \( 1 \leq s \leq d \),

\[
\sum_{l=1}^{d} \left( \lambda^{lk}_s \lambda^{ij}_l + \lambda^{li}_s \lambda^{jk}_l + \lambda^{lj}_s \lambda^{ki}_l \right) = 0.
\]
Example 4.11

Let \( A = k[x, y] \). Then there is an one-to-one correspondence between the co-Poisson Hopf structures \( q \) on \( A \) and \((I_x, I_y) \in \mathcal{I} \times \mathcal{I}\), given by

\[
q \mapsto (q(x), q(y)), \quad \text{and}
\]

\[
(l_x, l_y) \mapsto q : x^n y^m \mapsto nl_x \Delta(x^{n-1}y^m) + ml_y \Delta(x^n y^{m-1}).
\]
Theorem 4.12

Suppose $\text{char } k = 0$. Let $\hat{A} = k[[x_1, x_2, \cdots, x_d]]$ be the algebra of formal power series and $A = k[x_1, \cdots, x_d]$. Then there is an one-to-one corresponding between Poisson algebra structures on $\hat{A}$ and co-Poisson coalgebra structures on $A$. 

Q. -S. Wu

Co-Poisson coalgebras and (co-)Poisson Hopf algebras
Theorem 4.13

*There is a one-to-one corresponding between Poisson Hopf structures on $A$ and co-Poisson Hopf structures on $A*. More precisely, assume

$$\{x_i, x_j\} = \lambda^{ij}_1 x_1 + \cdots + \lambda^{ij}_d x_d$$

defines a Poisson Hopf structure on $A$. Let

$$I(x_s) = \sum_{1 \leq i, j \leq d} \lambda^{ij}_s x_i \otimes x_j$$

for $1 \leq s \leq d$ and $I(a) = 0$ for all other $a \in \mathcal{H}(A)$. Then $q(a) = I(a_1)\Delta(a_2)$ defines a co-Poisson Hopf structure on $A$. 
Motivation

Poisson structures and copoison structures
Duality between Poisson and co-Poisson structures

(Co-) Poisson structures on $k[x_1, \cdots, x_d]$

Motivation
Poisson structures and copoisson structures
Duality between Poisson and co-Poisson structures
(Co-) Poisson structures on $k[x_1, \cdots, x_d]$

Q. Lou, Q.-S. Wu, Co-Poisson structures on polynomial Hopf algebras, to appear in Sciences China, Mathematics
(arXiv:1601.04269v2 [math.RA])
Thank You!