One-Dimensional Line Schemes

Michaela Vancliff

University of Texas at Arlington, USA

http://www.uta.edu/math/vancliff/R  vancliff@uta.edu

Partial support from NSF DMS-1302050.
Motivation

Throughout, $\mathbb{k}$ = algebraically closed field.

Van den Bergh (early 1990s):
any quadratic algebra on 4 generators with 6 generic defining relations has 20 nonisomorphic truncated point modules of length 3 (20 is counted with multiplicity);
if also AS-regular, then it has a 1-parameter family of line modules (in today’s language, a 1-dimensional line scheme).

Shelton & Vancliff (late 1990s):
any quadratic algebra on 4 generators with 6 defining relations that

- has a finite scheme $\mathcal{Z}$ of truncated point modules of length 3 $\implies$ can recover the defining relations from $\mathcal{Z}$;
- or
- is AS-regular ( + a few more hyps ) & has a 1-diml line scheme $\mathcal{L}$ $\implies$ can recover the defining relations from $\mathcal{L}$. 
Longterm Goal
Classify all quadratic AS-regular algebras $A$ of gldim 4 using $\mathcal{Z}$ or $\mathcal{L}$.

Subgoal
Identify those $\mathcal{L}$ of dim = 1 where $|\mathcal{Z}| = 20$ (or $|\mathcal{Z}| < \infty$).

Regarding the subgoal: for an embedding in some “appropriate” projective space, can one determine possible degree(s) of $\mathcal{L}$, at least if $\dim(\mathcal{L}) = 1$?
I plan to address this question in today’s talk.

Notation
Write $A = T(V)/\langle R \rangle$ where
$\dim(V) = 4$, $R \subset V \otimes V$, $\dim(R) = 6$. 
Set-up (Points)

\[ \mathbb{P}(V^*) \times \mathbb{P}(V^*) \cong \Omega_1 = \mathbb{P}(\text{rank-1 elements}) \subset \mathbb{P}(V^* \otimes V^*) \]

\[ \text{dim} = 6 \]
\[ \text{deg} = 20 \]

\[ \mathbb{P}(R^\perp) \subset \mathbb{P}(V^* \otimes V^*) \]

\[ \mathfrak{z} \cong \Omega_1 \cap \mathbb{P}(R^\perp) \]

\[ \dim(\Omega_1 \cap \mathbb{P}(R^\perp)) \geq 6 + 9 - 15 = 0 \implies \mathfrak{z} \text{ nonempty.} \]

Also, \[ \deg(\Omega_1 \cap \mathbb{P}(R^\perp)) = (20)(1) = 20 \] by Bézout’s Thm

and examples of \( \mathfrak{z} \) are known where \( |\mathfrak{z}| < \infty \).

\[ \implies |\text{generic} \ \mathfrak{z}| = 20 \ (\text{counted with multiplicity}). \]

So, “generic” \( R \) means \( \mathbb{P}(R^\perp) \) meets \( \Omega_1 \) with minimal dimension,
in which case \( |\mathfrak{z}| = 20 \ (\text{counted with multiplicity}). \)
Set-up (Lines)

\[
\begin{align*}
\dim &= 11 \\
\mathbb{P}^5 &\cong \mathbb{P}(R) \subset \mathbb{P}(V \otimes V) \\
\widehat{\Omega}_2 &= \mathbb{P}(\text{rank } \leq 2 \text{ elements}) \subset \mathbb{P}(V \otimes V)
\end{align*}
\]

AS-regular etc \(\Rightarrow\) use Prop 2.8 in T. Levasseur & S.P. Smith’s paper

\(\Rightarrow\) line scheme \(\mathcal{L}_R \cong \Omega_2 \cap \mathbb{P}(R) \nexists\) any rank-1 elements.

\(\Rightarrow\) \(\dim(\text{each irred component of } \mathcal{L}_R) \geq 11 + 5 - 15 = 1.\)

Snag: in order to compare the lines with the points, we wish to view \(\mathcal{L}_R \subset \text{Grassmannian } G(2, V^*) = \text{scheme that parametrizes all lines in } \mathbb{P}(V^*).\)

As \(G(2, V^*) \overset{\text{Plücker}}{\longrightarrow} \mathbb{P}(\wedge^2 V^*) = \mathbb{P}^5,\) we have

**Question:** what is \(\deg(\mathcal{L}_R)\) viewed in this \(\mathbb{P}^5,\) at least when \(\dim(\mathcal{L}_R) = 1?\)

Joint work with A. Chirvasitu & S.P. Smith \(\Rightarrow\) \(\deg(\mathcal{L}_R) = 20\) if \(\dim(\mathcal{L}_R) = 1.\)
Approach

Main idea: work with “dual” scheme that has same degree as $\mathcal{L}_R$ in a “dual” $\mathbb{P}^5$.

i.e., let $\mathcal{L}_R^\perp$ = scheme in $G(2, V)$ that parametrizes all dim-2 $Q \subset V$ such that $(Q \otimes V) \cap R \neq 0$.

rank-2 element in $\mathbb{P}(V \otimes V)$

$$\mathcal{L}_R \cong \mathcal{L}_R^\perp \subset G(2, V) \xrightarrow{\text{Plücker}} \mathbb{P}(\wedge^2 V) = \mathbb{P}^5$$

& the map $\mathbb{P}(\wedge^2 V) \to \mathbb{P}(\wedge^2 V^*)$ is homogeneous of degree 1, so

$$\deg(\mathcal{L}_R^\perp) = \deg(\mathcal{L}_R).$$
Main Result

Theorem [Chirvasitu, Smith, V]

Let \( V \) be a 4-dimensional vector space, \( R \subset V \otimes V \), where \( \dim(R) = 6 \), & let \( \mathcal{L}_R \) be the scheme whose reduced variety is

\[
\{ Q \in G(2, V) : (Q \otimes V) \cap R \neq 0 \}.
\]

(Note: no hypothesis on \( T(V)/\langle R \rangle \) being regular or having good Hilbert series, etc.)

(a) \( \dim(\) each irred component of \( \mathcal{L}_R \) \( ) \geq 1\);

(b) \( \{ \mathcal{L}_R : R \subset V \otimes V, \dim(R) = 6, \dim(\mathcal{L}_R) = 1 \} \) is a flat family;

(c) if \( \text{char}(k) \neq 2 \) & if \( \dim(\mathcal{L}_R) = 1 \), then \( \deg(\mathcal{L}_R) = 20 \), where \( \mathcal{L}_R \hookrightarrow \mathbb{P}(\wedge^2 V) = \mathbb{P}^5 \).

The lack of homological hypotheses means the theorem is a result about 6-dimensional subspaces of the space of \( 4 \times 4 \) matrices.

(And when the algebra is not regular, the schemes \( \mathcal{L}_R \) and \( \mathcal{L}_R \) parametrize the truncated right line modules of dimension three.)
Idea of Proof

- Prove (a) and (b), and then use (b) (i.e., flatness) to prove deg($\mathcal{L}_R^\perp$) is a constant for all $R$ that satisfy the hypotheses of the theorem where $\dim(\mathcal{L}_R^\perp) = 1$;
- then exhibit an example that has deg($\mathcal{L}_R^\perp$) = 20.

For the example, we used an algebra I had previously studied with R. Chandler, but that work assumed $\text{char}(\mathbb{k}) = 0$. So we computed the degree assuming $\text{char}(\mathbb{k}) \neq 2$.

Will now present 3 examples of $\mathcal{L}_R$, where $T(V)/\langle R \rangle$ is regular & $\text{char}(\mathbb{k}) = 0$. 
1st Example [Chandler, V]

Let $\gamma, i \in \mathbb{k}^\times$, $i^2 = -1$, $V = \text{span of } x_1, \ldots, x_4$, & $R = \text{span of}$:

- $x_4x_1 - ix_1x_4$,  
- $x_2^2 - x_1^2$,  
- $x_3x_1 - x_1x_3 + x_2^2$,  
- $x_3x_2 - ix_2x_3$,  
- $x_4^2 - x_2^2$,  
- $x_4x_2 - x_2x_4 + \gamma x_1^2$.

If $\gamma(\gamma^2 - 4) \neq 0$, then $|\mathfrak{z}| = 20$ and can be grouped naturally in $\mathbb{P}^3$ into 2 sets of 2 points and 4 sets of 4 points (call latter type generic points).

If $\gamma(\gamma^2 - 16) \neq 0$, then $\mathfrak{L}_R = \text{union of 6 subschemes in } \mathbb{P}^5$:

- 1 nonplanar deg-4 elliptic curve in a $\mathbb{P}^3$ (i.e., spatial elliptic curve),
- 4 planar elliptic curves,
- a subscheme in a $\mathbb{P}^3$ consisting of the union of 2 nonsingular conics.

Each of the 16 generic points of $\mathfrak{z}$ lies on 6 distinct lines parametrized by $\mathfrak{L}_R$ (1 line from each of the above 6 subschemes).
Each of the remaining 4 points lies on infinitely many lines parametrized by $\mathfrak{L}_R$. 
2nd Example [Derek Tomlin, V]

Let $\alpha \in k$, $\alpha(\alpha^2 - 1) \neq 0$, $V = \text{span of } x_1, \ldots, x_4$, $R = \text{span of:}$

\[
\begin{align*}
    x_1x_3 + x_3x_1, & \quad x_2x_3 - x_3x_2, & \quad x_2x_4 + x_4x_2 - x_3^2, \\
    x_1x_4 + x_4x_1, & \quad x_2^2 - x_4^2, & \quad 2x_2^2 + \alpha x_3^2 - x_1^2.
\end{align*}
\]

$|\mathfrak{Z}| = 20$ & the points can be grouped naturally in $\mathbb{P}^3$ into 10 sets of 2 points.

$\mathcal{L}_R = \text{union of 6 subschemes in } \mathbb{P}^5$: 
- 1 nonplanar deg-4 elliptic curve in a $\mathbb{P}^3$, (i.e., spatial elliptic curve),
- 1 nonplanar deg-4 rational curve (with 1 singular point) in a $\mathbb{P}^3$,
- 2 planar elliptic curves,
- 2 subschemes, each of which consists of the union of a nonsingular conic and a line (that meets the conic in 2 distinct points).

$\exists$ 16 points $p \in \mathfrak{Z}$ such that $p$ lies on 6 lines (ctd with mult) of those parametrized by $\mathcal{L}_R$ (1 line from each of the above 6 subschemes, but some lines belong to more than 1 family). Each of the remaining 4 points lies on infinitely many lines parametrized by $\mathcal{L}_R$. 
3rd Example [Chirvasitu, Smith, Derek Tomlin]

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{k} \setminus \{0, -1, 1\}$, where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1 \alpha_2 \alpha_3 = 0$,

$V = \text{span of } x_1, \ldots, x_4$, $R = \text{span of:}$

$x_4 x_i - x_i x_4 - \alpha_i(x_j x_k - x_k x_j), \quad x_4 x_i + x_i x_4 - \alpha_i(x_j x_k + x_k x_j),$

where $(i, j, k)$ cycles through $(1, 2, 3)$.

$|\mathfrak{Z}| = 20 \& \text{the points can be grouped naturally in } \mathbb{P}^3 \text{ into 6 sets of 2 points and 8 other points.}$

$\mathcal{L}_R = \text{union of 7 irreducible subschemes in } \mathbb{P}^5:$

- 3 nonplanar deg-4 elliptic curves in a $\mathbb{P}^3$, (i.e., spatial elliptic curves $E_1, E_2, E_3$),
- 4 nonsingular conics.

$\exists 16 \text{ points } p \in \mathfrak{Z} \text{ such that } p \in 3 \text{ lines, each given by a distinct conic, and } p \in 3 \text{ other lines, each given by a distinct elliptic curve.}$

$\text{The remaining 4 points } p \in \mathfrak{Z} \text{ are such that } p \in 2 \text{ lines given by } E_i, i = 1, 2, 3 \text{ (6 total).}$

1st 2 examples are graded skew Clifford algebras, but 3rd does not appear to be related to a graded skew Clifford algebra.
Conjectures

Lines Conjecture
One of the generic classes of quadratic AS-regular algebra of gldim 4 has line scheme that consists of

- the union of 2 deg-4 spatial elliptic curves & 4 planar elliptic curves
  \[( 2)(4) + (4)(3) = 20 \];

and another generic class has line scheme that consists of

- the union of 4 deg-4 spatial elliptic curves & 2 nonsingular conics
  \[( 4)(4) + (2)(2) = 20 \).

Points Conjecture
Generic \( R \iff a \otimes b \in R^\perp \) iff \( b \otimes a \in R^\perp \).
Future Work?
Identify more $\mathcal{L}_R$, perhaps where $|z| < 20$?

Can $\mathcal{L}_R$ = a union of lines? 20 distinct lines? (even if $A$ not AS-regular)


- D. Tomlin & M. Vancliff, The One-Dimensional Line Scheme of a Family of Quadratic Quantum $\mathbb{P}^3$s, preprint 2017. (arXiv:1705.10426)

Appendix: \( G(2, V) \xrightarrow{\text{Pl"ucker}} \mathbb{P}(\wedge^2 V) = \mathbb{P}^5 \)

Write \( V = \bigoplus_{i=1}^{4} kx_i, \quad u = \sum_{i=1}^{4} u_ix_i, \quad w = \sum_{i=1}^{4} w_ix_i \in V, \)

\[
Q = k\ u \oplus k\ w \mapsto u \wedge w = \sum_{i<j} N_{ij} x_i \wedge x_j \in \mathbb{P}(\wedge^2 V) = \mathbb{P}^5
\]

& the \( N_{ij} \) are the \( 2 \times 2 \) minors of \[
\begin{bmatrix}
u_1 & u_2 & u_3 & u_4 \\
w_1 & w_2 & w_3 & w_4
\end{bmatrix}.
\]

The 6 \( N_{ij} \) are homogeneous coordinates on \( \mathbb{P}(\wedge^2 V) = \mathbb{P}^5. \)