NOTES FROM TALK AT ALGEBRA EXTRAVAGANZA! JULY 2017, TEMPLE UNIVERSITY

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This talk is mostly on joint work with Calin Chindris of U. Missouri [CK17], and the last theorem is joint work with Andrew T. Carroll of Depaul, Chindris, and Jerzy Weyman of U. Connecticut [CCKW17]. More detailed references and background can be found in our cited preprints.

1. Motivation

Throughout, K is a field and A a finite-dimensional (associative) K-algebra.

Central Problem: Classify finite-dimensional representations of A, up to isomorphism.

For most A, it seems that no list or algorithm to solve this problem is possible (this can be made precise using model theory, see work of M. Prest). Here is an alternative approach.

Geometric view: Construct algebraic varieties whose points parametrize isoclasses of representations, and study their geometric structure.

This has inherent limitations which can be made precise in the language of moduli problems, see Newstead's notes for example. Some intuition for these limitations can be developed through the following thought experiment which uses only linear algebra. Assume $K = \overline{K}$ and that the characteristic of K is zero from now on.

Motivating example/Informal exercise: Let $A = K[x]$ (infinite dimensional, but illustrates the main idea). Isomorphism classes of d-dimensional representations are in bijection with conjugacy classes of $d \times d$ matrices, so are classified by Jordan canonical form.
(1) Understand why no algebraic variety can "continuously" parametrize all these isoclasses, for $d \geq 2$. In scheme-theoretic language, you will always end up with a "non-separated" space because different conjugacy classes can have the same characteristic polynomial.
(2) Instead, parametrize isoclasses of a *dense subset* of the matrix space. For example, matrices with distinct eigenvalues are dense, and these isoclasses are parametrized by K^d/S_d (unordered d-tuples of distinct elements of $K-$ this is an algebraic variety). There are other solutions, for example using rational canonical form.

Another example appeared in Kenny Brown's talk: the natural parametrization of isoclasses of simple reps in his setup leads to a non-separated scheme, but if one restricts to the Azumaya locus, one gets a nice parametrization of an open, dense subset.

2. Moduli spaces of representations

The category of finite-dimensional representations of any A as in our assumptions is equivalent to the category of representations of a quotient of a path algebra of a quiver, $A = KQ/I$. So without loss of generality, we can take $A = KQ/I$. Let $\mathbf{d}$ be a dimension vector for Q; we can consider $\mathbf{d} \in K_0(A)$, the Grothendieck group of A. This gives rise to an affine algebraic variety $\operatorname{rep}(A, \mathbf{d})$ which parametrizes reps of class $\mathbf{d}$ along with a choice of basis (the analogue of the matrix space in the motivating example above). It is a closed subvariety of a product of matrix spaces.

Remarks: (1) The choice of basis means that isoclasses are in bijection with orbits in $\operatorname{rep}(A, \mathbf{d})$ under some group, not points. So this is not a variety which "classifies" representations.
(2) In contrast to the matrix space in the motivating example, $\operatorname{rep}(A, \mathbf{d})$ may not be irreducible and can have essentially any kind of singularities (as A and $\mathbf{d}$ vary).

To get rid of the choice of basis, we need to restrict to some subset of representations; the concept of *semistability* is the analogue of restricting to operators with distinct eigenvalues. Let $\theta \in \operatorname{Hom}_{\mathbb{Z}}(K_0(A), \mathbb{Z})$ be a *weight*, approximately equivalent to the concept of a *stability condition.*

Definition 2.1. Given θ as above, define the full subcategory of *θ-semistable representations* of A by

$$\operatorname{rep}(A)^{ss}_\theta = \{M \operatorname{rep}(A) \mid \theta([M]) = 0, \quad \text{and } \forall N \leq M : \ \theta([N]) \leq 0\}.$$

This is an abelian subcategory of $\operatorname{rep}(A)$. Let $\operatorname{rep}(A)^s_\theta$ be the colletion of simple objects of $\operatorname{rep}(A)^{ss}_\theta$, whose objects are called *θ-stable representations of A.* □

Geometrically, for each $\mathbf{d}$ the inclusions $\operatorname{rep}(A, \mathbf{d})^s_\theta \subseteq \operatorname{rep}(A, \mathbf{d})^{ss}_\theta \subset \operatorname{rep}(A, \mathbf{d})$ are both open, and thus these subsets are dense *when nonempty.* Mumford's geometric invariant theory (GIT) gives a quotient morphism $\operatorname{rep}(A, \mathbf{d})^{ss}_\theta \twoheadrightarrow \mathcal{M}(A, \mathbf{d})^{ss}_\theta$, whose target is a projective variety known as the *moduli space of θ-semistable representations of dimension vector* $\mathbf{d}$. Its points are in bijection with certain *isoclasses* of representations of A, namely the ones of class $\mathbf{d}$ which are semisimple in $\operatorname{rep}(A)^{ss}_\theta$. We want to study the structure of these varieties, but we must be careful what we study, since any projective variety whatsoever can be realized as $\mathcal{M}(A, \mathbf{d})^{ss}_\theta$ for some $A, \mathbf{d}, \theta$.

3. Results

3.1. **A decomposition theorem.** Assume from now on that $C \subseteq \operatorname{rep}(A, \mathbf{d})$ is a closed subvariety such that $\mathcal{M}(C)^{ss}_\theta$ is an irreducible component of $\mathcal{M}(A, \mathbf{d})^{ss}_\theta$ (all components are indeed of this form, so this is no loss of generality).

Definition 3.1. A collection $(C_i \subseteq \operatorname{rep}(A, \mathbf{d}_i), m_i \in \mathbb{Z}_{>0})_{i=1}^r$ is a θ-stable decomposition of C if there is a dense subset of θ-semistable $M \in C$ such that $\operatorname{gr}_\theta(M)$ (the associated graded object in the category $\operatorname{rep}(A)^{ss}_\theta$) has exactly m_i θ-stable summands from each C_i. □

It is a fact that a θ-stable decomposition exists for C as in our assumption. We can think of such decomposition as geometric Jordan-Holder factors for C (w.r.t. θ). The main theorem of [CK17] says that these give a lof of info about the geometry of moduli spaces.

Theorem 3.2 (joint with Chindris)**.** *For any finite-dimensional $A, \mathbf{d}, \theta, C$ as above, there exists a finite, birational morphism*

$$\Psi : S^{m_1}\mathcal{M}(C_1)^{ss}_\theta \times \cdots \times S^{m_r}\mathcal{M}(C_r)^{ss}_\theta \to \mathcal{M}(C)^{ss}_\theta$$

"decomposing" the right hand side. If $\mathcal{M}(C)^{ss}_\theta$ is a normal variety, then Ψ is an isomorphism.

Here, for a variety Z, we write $S^k Z = (\prod_{i=1}^k Z)/S_k$ for the k-th symmetric power of Z. At the level of underlying sets, the statement is almost obvious once properly understood, so the main content of the theorem is that the map is a morphism of varieties. When $A = KQ$, each $\mathcal{M}(C)^{ss}_{\theta}$ is always normal; the isomorphism in this case is essentially equivalent to a theorem of Derksen and Weyman from 2006.

3.2. Application to moduli of tame algebras.

Definition 3.3. An algebra A is *tame* if, for each fixed d, almost all isomorphism classes of d-dimensional indecomposables fall into finitely many one K parameter families. □

For example, $A = K[x]$ is tame, as are path algebras of quivers of affine Dynkin type (see James Zhang's talk). Although any projective variety can be realized as a $\mathcal{M}(C)^{ss}_{\theta}$ if we allow A to vary over all algebras, there are a number of results in the literature showing that moduli spaces associated to tame algebras tend to be nicer. The following corollary of our decomposition theorem above makes this more precise.

Corollary 3.4. *Every $\mathcal{M}(C)^{ss}_{\theta}$ for a tame algebra is birational to some $\mathbb{P}^N$ (i.e. a* rational variety*).*

Special biserial algebras are a prominent class of tame algebras whose indecomposables admit a nice combinatorial description. They arise naturally in modular group representation theory and categorification of cluster algebras from surfaces, for example. We were able to completely determine isomorphism types of moduli spaces of special biserial algebras, and it turns out they are as nice as possible.

Theorem 3.5 (joint with Carroll, Chindris, Weyman)**.** *Every $\mathcal{M}(C)^{ss}_{\theta}$ for a special biserial algebra is isomorphic to $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_r}$ for some positive integers m_i.*

Proving this theorem involves reducing to some nice subset of C using another theorem from [CK17] (which was not mentioned in this talk), then showing these C are normal using work of Lusztig connecting them with affine Schubert varieties. The isomorphism of the theorem is the one of the decomposition theorem above.

References

[CCKW17] Andrew T. Carroll, Calin Chindris, Ryan Kinser, and Jerzy Weyman, *Moduli spaces of representations of special biserial algebras*, `arxiv:1706.06022`, 2017.

[CK17] Calin Chindris and Ryan Kinser, *Decomposing moduli of representations of finite-dimensional algebras*, `arxiv:1705.10255`, 2017.

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