# Detecting uniparametricality and polynomial stability of graded algebras via cocycle twist invariants

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**Abstract.** We discuss how various graded algebras can and cannot be obtained from uniparameter ones by twisting, introduce some new invariants to help detect the difference, and describe additional invariants obtained from expansion of grading groups. In particular, we show that a number of quantum algebras are "truly multiparameter", in that they cannot be written as cocycle twists of uniparameter algebras – algebras for which the scalars in the relevant commutation relations are powers of a fixed scalar. The ideas apply to Z-graded algebras by expanding the grading group from Z to the character group of a maximal torus of an appropriate automorphism group. We will also discuss settings in which our invariants are stable under passage to polynomial rings, thus allowing one to show that certain algebras cannot become isomorphic even after making polynomial extensions. This is joint work with Milen Yakimov [3].

Many algebras in the "quantum world" come in both uniparameter and multiparameter flavors, the first meaning that the essential parameters used in the relations of the algebra are powers of a fixed scalar.

$K$ = a base field.

Two standard examples:

- Quantum affine spaces $\mathcal{O}_q(K^n)$, $\mathcal{O}_{\boldsymbol{q}}(K^n)$

  Generators: $x_1, \dots, x_n$

  Standard uniparameter relations: $x_i x_j = q x_j x_i \;\; \forall\, i < j \;\; (q \in K^\times)$

  Multiparameter relations: $x_i x_j = q_{ij} x_j x_i \;\; \forall\, i, j \quad \big(\boldsymbol{q} = (q_{ij}) \in M_n(K^\times)$ multiplicatively skew-symmetric$\big)$

- Quantum matrix algebras $\mathcal{O}_q(M_n(K))$, $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(K))$

  Generators: $X_{ij}$, $i, j = 1, \dots, n$

  Standard uniparameter relations:

$$\begin{tikzcd} X_{ij} \arrow[r, "q"] \arrow[d, "q"'] & X_{im} \arrow[d, "q"] \arrow[dl, "1"'] \\ X_{lj} \arrow[r, "q"'] & X_{lm} \end{tikzcd} \qquad X_{ij}X_{lm} - X_{lm}X_{ij} = (q - q^{-1}) X_{im} X_{lj} \quad (q \in K^\times)$$

(Here $x \xrightarrow{p} y$ records a commutation relation $xy = pyx$.)

Multiparameter relations:

$$\begin{tikzcd}
X_{ij} \arrow[r, "p_{mj}"] & X_{im} \\
X_{lj} \arrow[u, "\lambda p_{li}"] \arrow[ur, "\lambda p_{li} p_{mj}" description] \arrow[r, "p_{mj}"'] & X_{lm} \arrow[u, "\lambda p_{li}"']
\end{tikzcd}$$

$$X_{lm}X_{ij} - p_{li}p_{jm}X_{ij}X_{lm} = (\lambda - 1)p_{li}X_{im}X_{lj}$$

$$(\lambda \in K^\times, \quad \boldsymbol{p} = (p_{ij}) \in M_n(K^\times)$$
multiplicatively skew-symmetric)

The parameters in quantum affine spaces can all be accounted for by twisting with cocycles. In quantum matrices, it's possible to account for all but one parameter that way.

We describe some of the story, and various invariants it led to.

**Cocycle Twists**

$\Gamma =$ an abelian group (additive)

A 2-*cocycle* $\in Z^2(\Gamma, K^\times)$ is a map $c : \Gamma \times \Gamma \to K^\times$ such that

$$c(s, t+u)c(t, u) = c(s+t, u)c(s, t), \quad \forall\, s, t, u \in \Gamma.$$

Suppose $A$ is a $\Gamma$-graded $K$-algebra.

The *twist of* $A$ *by* $c$ is a $\Gamma$-graded $K$-algebra $A^c$ on the vector space $A$, with multiplication $*_c$ such that

$$x *_c y = c(s,t)\, xy, \quad \forall\, x \in A_s\,,\ y \in A_t\,,\ s, t \in \Gamma.$$

**Observation/Exercise.** Grade $\mathcal{O}_{\boldsymbol{q}}(K^n)$ by $\Gamma = \mathbb{Z}^n$ with $\deg x_i = e_i$ (the $i$ th standard basis element of $\mathbb{Z}^n$).

Then $\mathcal{O}_{\boldsymbol{q}}(K^n) = K[x_1, \dots, x_n]^c$ for a suitable $c$.

**Prop.** [Artin-Schelter-Tate, Sudbery] Grade $A = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(K))$ by $\Gamma = \mathbb{Z}^n \times \mathbb{Z}^n$, with $\deg X_{ij} = (e_i, e_j)$.

Then $A = \mathcal{O}_{\lambda,\mathbf{1}}(M_n(K))^c$ for a suitable $c$.

There's no obvious way to obtain quantum matrix algebras as cocycle twists of commutative algebras. It can be verified that this is, in fact, impossible, by working out the pattern of all cocycle twists of the commutative polynomial algebra $\mathcal{O}(M_n(K))$ and showing that most quantum matrix algebras are not covered.

Better: Detect this by some invariant.

Notice: The key scalars in commutation relations all appear as commutators in quotient division rings. In particular:

$$zw = qwz \implies zwz^{-1}w^{-1} = q$$

Focusing on such scalars leads to an invariant that Alev and Dumas used to separate quantum planes and their division rings. We extend and adapt their invariant.

- **Throughout:** $A =$ a semiprime Goldie ring, with quotient ring $\operatorname{Fract} A$.

**The Alev-Dumas and Γ-Twist Invariants**

**Def.** $AD(A) := \underbrace{GL_1(\operatorname{Fract} A)^1}_{\text{commutator subgroup}} \cap K^\times$

**Prop.** [Alev-Dumas] $AD(\mathcal{O}_{\boldsymbol{q}}(K^n)) = \langle q_{ij} \rangle$

The AD-invariant is obviously affected by cocycle twists. Consider all the subgroups of $K^\times$ that occur as AD-invariants of cocycle twists of a given algebra, then intersect them to get an invariant unaffected by cocycle twists.

**Def.** For Γ-graded $A$, set $\operatorname{tw}_\Gamma(A) := \bigcap_{c \in Z^2(\Gamma, K^\times)} AD(A^c)$.

**Thm.** $\operatorname{tw}_{\mathbb{Z}^n}(\mathcal{O}_{\boldsymbol{q}}(K^n)) = \langle 1 \rangle$
$AD(\mathcal{O}_q(M_n(K)) = \langle q \rangle$ and $\operatorname{tw}_{\mathbb{Z}^n \times \mathbb{Z}^n}(\mathcal{O}_q(M_n(K)) = \langle q^2 \rangle$
$AD(\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(K)) = \langle \lambda, p_{ij} \rangle$ and $\operatorname{tw}_{\mathbb{Z}^n \times \mathbb{Z}^n}(\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(K)) = \langle \lambda \rangle$

**Cor.** $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(K))$ and $\mathcal{O}_q(M_n(K))$ are cocycle twists of $\mathcal{O}(M_n(K))$ only if $\lambda = 1$ or $q = \pm 1$.

The Artin-Schelter-Tate/Sudbery result shows that all the multiparametricality in quantum matrix algebras arises from cocycle twists. One could say these algebras are "essentially uniparameter".

Other quantum algebras are "genuinely multiparameter". For example:

- Quantized Weyl algebras $A_n^{Q,P}(K)$
  $Q = (q_i) \in (K^\times)^n$, $P = (p_{ij}) \in M_n(K^\times)$ multiplicatively skew-symmetric.
  Generators: $x_1, y_1, \dots, x_n, y_n$
  Relations:

$$y_i y_j = p_{ij} y_j y_i \qquad\qquad x_i x_j = q_i p_{ij} x_j x_i \quad (i < j)$$

$$x_i y_j = \begin{cases} p_{ji} y_j x_i & (i < j) \\ q_j p_{ji} y_j x_i & (i > j) \\ 1 + q_j y_j x_j + \sum_{l<j} (q_l - 1) y_l x_l & (i = j) \end{cases}$$

Grade $A_n^{Q,P}(K)$ by $\Gamma = \mathbb{Z}^n$, with $\deg x_i = e_i$ and $\deg y_i = -e_i$.

**Thm.** $\operatorname{tw}_\Gamma(A_n^{Q,P}(K)) = \langle q_1, \dots, q_n \rangle$

**Cor.** If $\langle q_1, \dots, q_n \rangle$ is not cyclic, $A_n^{Q,P}(K)$ cannot be a cocycle twist of any uniparameter Γ-graded algebra.

The above ideas don't appear to hold much for $\mathbb{Z}$-graded algebras, since 2-cocycles on $\mathbb{Z}$ are rather restricted. In particular:

**Exer.** $c \in Z^2(\mathbb{Z}, K^\times) \implies c(s,t) = c(t,s) \quad \forall\, s,t \in \mathbb{Z}$.

So if $A$ is $\mathbb{Z}$-graded, $x, y \in A$ homogeneous, and $xy = qyx +$ (other terms), then $x *_c y = qy *_c x +$ (other terms) in any $A^c$.

**E.G.** If $A = \mathcal{O}_{\boldsymbol{q}}(K^n)$ is $\mathbb{Z}$-graded with $\deg x_i = 1$ for all $i$, then $A^c \cong A$ for all $c \in Z^2(\mathbb{Z}, K^\times)$. Thus, $\mathrm{tw}_{\mathbb{Z}}(A) = \langle q_{ij} \rangle$.

This ignores the fact that $\mathcal{O}_{\boldsymbol{q}}(K^n)$ is "really" $\mathbb{Z}^n$-graded. The question is, how can we see a $\mathbb{Z}^n$-grading within the context of a $\mathbb{Z}$-graded algebra? This can be done via connections with torus actions.

**Torus Actions**

An *algebraic $K$-torus* is any affine algebraic group $H \cong$ some $K^n$.

The (rational) *character group* of $H$ is $X(H) := \{\text{rational characters } H \to K^\times\}$, with pointwise multiplication.

If $H \cong K^n$, then $X(H) \cong \mathbb{Z}^n$ (with the natural projection maps as basis).

**Thm.** Every affine algebraic group has maximal tori, and they are all conjugate.

An action of $H$ on $A$ is *rational* if

(1) $A$ is a directed union of finite dimensional $H$-stable subspaces $V_i$;
(2) The restriction maps $H \to GL(V_i)$ are morphisms of algebraic groups.

**Prop.** Rational actions of $H$ on $A$ by $K$-algebra automorphisms are equivalent to $X(H)$-gradings.

Plan: Given a $K$-algebra $A$, find a group of automorphisms which is an algebraic group, take a maximal torus $H$, give $A$ the $X(H)$-grading. Then we can take the twist invariant of $A$ relative to this grading.

Various settings yield algebraic groups of automorphisms. For this talk, let's concentrate on graded automorphism groups.

Let $A = \bigoplus_{n \geq 0} A_n$ be nonnegatively graded, and set

$$\mathrm{Aut}_{\mathrm{gr}}\, A := \{\phi \in \mathrm{Aut}\, A \mid \phi(A_n) = A_n \;\forall\, n\}.$$

Assume that $A$ is a finitely generated algebra and that $\dim_K A_n < \infty \;\; \forall\, n$.

**Prop.** Choose $d$ such that $A$ is generated by $V_d := \bigoplus_{n=0}^{d} A_n$.

(a) Restriction to $V_d$ gives $\mathrm{Aut}_{\mathrm{gr}}\, A \cong$ a Zariski-closed subgroup of $GL(V_d)$.

(b) Via (a), $\mathrm{Aut}_{\mathrm{gr}}\, A$ becomes an affine algebraic group. This structure is independent of the choice of $d$.

(c) The action of $\mathrm{Aut}_{\mathrm{gr}}\, A$ on $A$ is rational.

**A General Twist Invariant**

**Def./Prop.** $\mathrm{tw}(A) := \mathrm{tw}_{X(H)}(A)$, where $H$ is a maximal torus of $\mathrm{Aut}_{\mathrm{gr}} A$. This is independent of the choice of $H$.

**E.G.** (a) Grade $A = \mathcal{O}_{\boldsymbol{q}}(K^n)$ by $\mathbb{Z}$ with $\deg x_i = 1$.

$H = (K^\times)^n$ is ($\cong$) a maximal torus of $\mathrm{Aut}_{\mathrm{gr}} A$, and the grading of $A$ by $X(H) \cong \mathbb{Z}^n$ is the standard one.

Therefore $\mathrm{tw}(A) = \mathrm{tw}_{\mathbb{Z}^n}(A) = \langle 1 \rangle$.

(b) Similarly, $\mathrm{tw}(\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(K)) = \langle \lambda \rangle$ and $\mathrm{tw}(\mathcal{O}_q(M_n(K)) = \langle q^2 \rangle$, where these algebras are $\mathbb{Z}_{\geq 0}$-graded with $\deg X_{ij} = 1 \;\; \forall\, i, j$.

**Polynomial Stability**

This term refers to invariance under polynomial extensions, namely the property that if polynomial rings over algebras $A$ and $B$ are isomorphic, then $A \cong B$. To test this, it's good to have stable invariants – invariants that remain the same for an algebra $A$ and a polynomial ring over $A$. The invariants discussed above satisfy this. It shouldn't seem too surprising for the AD-invariant, which measures commutators, since throwing in a commuting polynomial variable doesn't create new commutators in any obvious way.

As always, we work with semiprime Goldie $K$-algebras.

**Prop.** $AD(A[z]) = AD(A)$.

**Cor.** If $\boldsymbol{p} \in M_m(K^\times)$ and $\boldsymbol{q} \in M_n(K^\times)$ are multiplicatively skew-symmetric with $\langle p_{ij} \rangle \neq \langle q_{ij} \rangle$, then

$$\mathcal{O}_{\boldsymbol{p}}(K^m)[z_1, \dots, z_s] \not\cong \mathcal{O}_{\boldsymbol{q}}(K^n)[z_1, \dots, z_t] \quad \forall\, s, t.$$

**Thm.** Assume $A$ is $\Gamma$-graded. Grade $A[z]$ by $\Gamma \times \mathbb{Z}$ so that $\deg z = (0, 1)$.

Then $\mathrm{tw}_{\Gamma\times\mathbb{Z}}(A[z]) = \mathrm{tw}_\Gamma(A)$.

**Cor.** Grade $A = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(K))$ and $A' = \mathcal{O}_{\lambda',\boldsymbol{p}'}(M_n(K))$ by $\Gamma = \mathbb{Z}^n \times \mathbb{Z}^n$ in the standard way, and grade $A[z_1, \dots, z_s]$ and $A'[z_1, \dots, z_s]$ by $\Gamma \times \mathbb{Z}^s$ so that $\deg z_i = (0, e_i)$. If $\langle \lambda \rangle \neq \langle \lambda' \rangle$, then

$$A[z_1, \dots, z_s] \not\cong A'[z_1, \dots, z_s] \quad \text{as} \quad (\Gamma \times \mathbb{Z}^s)\text{-graded algebras.}$$

In fact, these algebras are not cocycle twists of each other.

The general twist invariant extends such non-isomorphisms.

**Thm.** Assume $A$ is finitely generated and nonnegatively graded, with $A_0 = K$ and $\dim A_n < \infty$ for all $n$. Grade $A[z]$ by $\mathbb{Z}$ with $\deg z = 1$.

Then $\mathrm{tw}(A[z]) = \mathrm{tw}(A)$.

**Thm.** Let $A = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(K))$ and $A' = \mathcal{O}_{\lambda',\boldsymbol{p}'}(M_{n'}(K))$. Grade $A[z_1, \dots, z_s]$ and $A'[z_1, \dots, z_t]$ by $\mathbb{Z}$ with $\deg X_{ij} = \deg z_i = 1$. If $\langle\lambda\rangle \neq \langle\lambda'\rangle$, then

$$A[z_1, \dots, z_s] \ncong A'[z_1, \dots, z_t] \quad \text{as} \quad \mathbb{Z}\text{-graded algebras.}$$

## References

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