Singularity categories of some noncommutative deformations

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Commutative singularities

- Work over $k = \mathbb{C}$.
- Commutative ring $R \rightsquigarrow$ Geometric object $\text{Spec } R$.
- In particular, we can study the singularities of $\text{Spec } R$.
- e.g. $R = \mathbb{C}[x, y, z]/(x^2 + y^2 - z^2)$

\[ \text{Spec } R = \]

- This is an example of an $\mathbb{A}_1$ singularity.
Background and motivation

Noncommutative singularities

▶ What about if $R$ is noncommutative?
▶ One issue: $\text{Spec } R = \{\text{prime ideals of } R\}$ is often too small.
▶ “Geometric properties of $\text{Spec } R$” $\iff$ “Algebraic properties of $R$”
▶ $\rightsquigarrow$ If $R$ is noncommutative, we say it has a geometric property if it has the corresponding algebraic property.
▶ Commutative fact: $\text{Spec } R$ is smooth $\iff$ $\text{gl.dim } R < \infty$.

Definition

A (possibly noncommutative) ring $R$ is singular (resp. smooth or nonsingular) if its global dimension is infinite (resp. finite).

▶ We would also like the be able to describe the singularities of $R$. 
Singularity categories

- Technical standing assumption: \( R \) is Gorenstein i.e. noetherian and \( \text{i.dim} \ R_R = \text{i.dim} \ R R < \infty \).
- In ’86, Buchweitz defined the singularity category of a noetherian ring \( R \) to be
  \[
  \mathcal{D}_{\text{sg}}(R) := \frac{\mathcal{D}^b(R)}{\text{Perf } R}.
  \]
  This is a triangulated category (with translation \( \Sigma \)).

**Lemma**

\( \mathcal{D}_{\text{sg}}(R) \) is trivial \( \iff \) \( R \) is smooth.

- Observation: don’t need \( R \) to be commutative.
- We can compare the singularities of two rings by comparing their singularity categories.
- “The bigger \( \mathcal{D}_{\text{sg}}(R) \), the more singular \( R \) is.”
Example: Kleinian singularities

- Family of surface singularities with coordinate rings $\mathbb{C}[u, v]^G \cong \mathbb{C}[x, y, z]/(f)$, where $G$ is a finite subgroup of $\text{SL}(2, \mathbb{C})$.
- Parametrised by simply laced Dynkin diagrams $Q$. Write $R_Q$ for the corresponding coordinate ring.
- Properties of $\mathcal{D}_{sg}(R_Q)$:
  - Krull-Schmidt category;
  - $\{\text{indecomposable objects}\} \leftrightarrow \{\text{vertices of } Q\}$;
  - $\Sigma$ induces a graph automorphism of $Q$.
- e.g. $R_{A_5}$

![Diagram of Dynkin diagram $A_5$ with vertices $V_1$ to $V_5$ connected by arrows representing the automorphism $\Sigma$.]
Deformations of Kleinian singularities

- In ’98, Crawley-Boevey and Holland introduced a family of deformations $\mathcal{O}^\lambda(\tilde{Q})$ of the $R_Q = \mathbb{C}[u, v]^G$.
- Require two pieces of data:
  - an extended Dynkin graph $\tilde{Q}$;
Deformations of Kleinian singularities

\[ \tilde{\mathbb{A}}_n \]

\[ \tilde{\mathbb{D}}_n, \ n \geq 4 \]

\[ \tilde{\mathbb{E}}_6 \]

\[ \tilde{\mathbb{E}}_7 \]

\[ \tilde{\mathbb{E}}_8 \]
In ’98, Crawley-Boevey and Holland introduced a family of deformations $O^\lambda(\tilde{Q})$ of the $R_Q = \mathbb{C}[u, v]^G$.

Require two pieces of data:
- an extended Dynkin graph $\tilde{Q}$;
- a weight $\lambda \in \mathbb{C}^{\tilde{Q}_0}$ (a complex number for each vertex of $\tilde{Q}$).

If we input a type $\tilde{Q}$ graph, we are deforming $R_Q$.

We need another definition before we can define the deformations.
Let $Q$ be a quiver without loops.

Form the double $\overline{Q}$ of $Q$ by adding a reverse arrow $\overline{\alpha} : j \to i$ for each arrow $\alpha : i \to j$.

Choose a weight $\lambda$.

Then the deformed preprojective algebra is $\Pi^\lambda(Q) = \mathbb{C}\overline{Q}/I$, where $I$ is the two-sided ideal with generators

$$\sum_{\alpha \in Q_1, t(\alpha) = i} \alpha \overline{\alpha} - \sum_{\alpha \in Q_1, h(\alpha) = i} \overline{\alpha} \alpha = \lambda_i e_i$$
Recall that $\mathcal{O}^\lambda(\tilde{Q})$ depends on the data of an extended Dynkin graph $\tilde{Q}$ and a weight $\lambda \in \mathbb{C}^{\tilde{Q}_0}$.

Choose any orientation for the edges of $\tilde{Q}$ to get a quiver $\tilde{Q}$. Then

$$\mathcal{O}^\lambda(\tilde{Q}) := e_0 \Pi^\lambda(\tilde{Q}) e_0.$$ 

Easy to write down a presentation in type $\mathbb{A}$:

$$\mathbb{C}\langle x, y, z \rangle \cong \mathbb{C}\langle x, y, z \rangle / \langle xz = (z + \sum_{i=0}^n \lambda_i)x, \ xy = \prod_{i=0}^n \left( z + \sum_{j=1}^i \lambda_j \right) \rangle,$$

$$yz = (z - \sum_{i=0}^n \lambda_i)y, \ yx = \prod_{i=0}^n \left( z - \sum_{j=1}^i \lambda_j \right) \rangle,$$

i.e., they’re examples of generalised Weyl algebras.
Some properties of $\mathcal{O}^\lambda(\tilde{Q})$

- Why are these deformations?

**Lemma (Crawley-Boevey – Holland)**

1. $\mathcal{O}^0(\tilde{Q}) \cong R_Q$.
2. There exists a filtration of $\mathcal{O}^\lambda(\tilde{Q})$ such that $\text{gr} \mathcal{O}^\lambda(\tilde{Q}) \cong R_Q$.

- It is also easy to detect when $\mathcal{O}^\lambda(\tilde{Q})$ is commutative:

**Lemma (Crawley-Boevey – Holland)**

$\mathcal{O}^\lambda(\tilde{A}_n)$ is commutative iff $\sum_{i=0}^{n} \lambda_i = 0$ (there are similar conditions for the other types).
Example

Consider $O^\lambda(\tilde{Q})$ for the following data:

Commutative

Noncommutative
The main result

The problem

Goal

Determine $\mathcal{D}_{sg}(\mathcal{O}^\lambda(\tilde{Q}))$.

- This is difficult for arbitrary $\lambda$. We can simplify matters, but first:

Definition

Call a weight quasi-dominant if $\lambda_i$ “lies in the right-half of the complex plane” for all $i \geq 1$. 
Simplifying the problem

**Lemma (Boddington – Levy)**

*Given a weight $\lambda$ for $\tilde{Q}$, there exists a quasi-dominant weight $\lambda'$ such that $\mathcal{O}^{\lambda}(\tilde{Q}) \cong \mathcal{O}^{\lambda'}(\tilde{Q})$.***

- Henceforth, assume all weights are quasi-dominant.
- Aside: there is an algorithm to find $\lambda'$.  

\[
egin{array}{ccc}
1 & 2 & -1 \\
0 & 3 & 1 \\
2 & 1 & -1 \\
0 & 3 & 1 \\
1 & 2 & 0
\end{array}
\]
Detecting smoothness

This makes it easy to detect smoothness:

Lemma (Crawley-Boevey – Holland)

$O^\times(\tilde{Q})$ is singular iff $\lambda_i = 0$ for some $i \neq 0$.

For example:

- Commutative and singular
- Noncommutative and singular
Detecting smoothness

This makes it easy to detect smoothness:

**Lemma (Crawley-Boevey – Holland)**

\( O^\lambda(\tilde{Q}) \) is singular iff \( \lambda_i = 0 \) for some \( i \neq 0 \).

For example:

- Commutative and smooth
- Noncommutative and smooth
The main result

Theorem (C., 2016)

Let $\tilde{Q}$ be an extended Dynkin graph, and let $\lambda$ be a quasi-dominant weight for $\tilde{Q}$. Write $Q_\lambda$ for the full subgraph of $\tilde{Q}$ obtained by removing

- vertex 0, and
- each vertex $i \geq 1$ with $\lambda_i \neq 0$.

Then $Q_\lambda = \bigcup_{i=1}^{r} Q_i$ is a disjoint union of Dynkin graphs, and there is a triangle equivalence

$$\mathcal{D}_{sg}(\mathcal{O}_\lambda(\tilde{Q})) \simeq \bigoplus_{i=1}^{r} \mathcal{D}_{sg}(R_{Q_i}).$$
Theorem (C., 2016)

Let $\tilde{Q} \in \{\tilde{A}, \tilde{D}, \tilde{E}\}$ and $\lambda$ be quasi-dominant. Let $Q_\lambda$ be the full subgraph of $\tilde{Q}$ obtained by removing vertex 0 and each vertex $i \geq 1$ with $\lambda_i \neq 0$. Then $Q_\lambda = \bigsqcup_{i=1}^{r} Q_i$, $(Q_i$ Dynkin), and $\mathcal{D}_{sg}(O^\lambda(\tilde{Q})) \simeq \bigoplus_{i=1}^{r} \mathcal{D}_{sg}(R_{Q_i})$.
Example: \( \tilde{Q} = \tilde{A}_5 \)

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\[\begin{array}{ccc}
1 & 1 & 4 \\
2 & 3 & 1 \\
1 & 4 & 1 \\
0 & 5 & 1 \\
-5 & 1 & \end{array}\]

$\mathcal{D}_{sg}(\mathcal{O}^\lambda(\tilde{Q})) \simeq \{0\}$
Example: \( \widetilde{Q} = \widetilde{A}_5 \)

**Theorem (C., 2016)**

Let \( \widetilde{Q} \in \{ \widetilde{A}, \widetilde{D}, \widetilde{E} \} \) and \( \lambda \) be quasi-dominant. Let \( Q_\lambda \) be the full subgraph of \( \widetilde{Q} \) obtained by removing vertex 0 and each vertex \( i \geq 1 \) with \( \lambda_i \neq 0 \). Then \( Q_\lambda = \bigsqcup_{i=1}^r Q_i \), \((Q_i \text{ Dynkin})\), and \( \mathcal{D}_{sg}(O^\lambda(\widetilde{Q})) \cong \bigoplus_{i=1}^r \mathcal{D}_{sg}(R_{Q_i}) \).
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\[ \mathcal{D}_{sg}(O^\lambda(\tilde{Q})) \simeq \mathcal{D}_{sg}(R_{A_1}) \oplus \mathcal{D}_{sg}(R_{A_3}) \]
The main result

Example: \( \tilde{Q} = \tilde{A}_5 \)

**Theorem (C., 2016)**

Let \( \tilde{Q} \in \{ \tilde{A}, \tilde{D}, \tilde{E} \} \) and \( \lambda \) be quasi-dominant. Let \( Q_\lambda \) be the full subgraph of \( \tilde{Q} \) obtained by removing vertex 0 and each vertex \( i \geq 1 \) with \( \lambda_i \neq 0 \). Then \( Q_\lambda = \bigsqcup_{i=1}^r Q_i, (Q_i \text{ Dynkin}) \), and \( \mathcal{D}_{sg}(O^\lambda(\tilde{Q})) \simeq \bigoplus_{i=1}^r \mathcal{D}_{sg}(R_{Q_i}) \).

\[
\begin{align*}
\tilde{Q} &= \tilde{A}_5 \\
\mathcal{D}_{sg}(O^\lambda(\tilde{Q})) &\simeq \mathcal{D}_{sg}(R_{A_1}) \oplus \mathcal{D}_{sg}(R_{A_3})
\end{align*}
\]
Example: $\tilde{Q} = \tilde{\mathcal{D}}_7$

Theorem (C., 2016)

Let $\tilde{Q} \in \{\tilde{A}, \tilde{D}, \tilde{E}\}$ and $\lambda$ be quasi-dominant. Let $Q_\lambda$ be the full subgraph of $\tilde{Q}$ obtained by removing vertex 0 and each vertex $i \geq 1$ with $\lambda_i \neq 0$. Then $Q_\lambda = \bigsqcup_{i=1}^{r} Q_i$, ($Q_i$ Dynkin), and $\mathcal{D}_{sg}(\mathcal{O}^\lambda(\tilde{Q})) \simeq \bigoplus_{i=1}^{r} \mathcal{D}_{sg}(\mathcal{R}_{Q_i})$. 

\[ \mathcal{D}_{sg}(\mathcal{O}^\lambda(\tilde{Q})) \simeq \mathcal{D}_{sg}(\mathcal{R}_{A_2}) \oplus \mathcal{D}_{sg}(\mathcal{R}_{D_4}) \]
The main result

Example:  \( \tilde{Q} = \tilde{\mathbb{D}}_7 \)

**Theorem (C., 2016)**

Let \( \tilde{Q} \in \{ \tilde{A}, \tilde{D}, \tilde{E} \} \) and \( \lambda \) be quasi-dominant. Let \( Q_{\lambda} \) be the full subgraph of \( \tilde{Q} \) obtained by removing vertex 0 and each vertex \( i \geq 1 \) with \( \lambda_i \neq 0 \). Then \( Q_{\lambda} = \bigsqcup_i^r Q_i, (Q_i \text{ Dynkin}), \) and \( \mathcal{D}_{\text{sg}}(O^\lambda(\tilde{Q})) \cong \bigoplus_i^r \mathcal{D}_{\text{sg}}(R_{Q_i}) \).

\[
\begin{align*}
\mathcal{D}_{\text{sg}}(O^\lambda(\tilde{Q})) & \cong \mathcal{D}_{\text{sg}}(R_{A_2}) \oplus \mathcal{D}_{\text{sg}}(R_{D_4}) \\
\end{align*}
\]
Some remarks

▶ Intuition: deforming a singularity should make it no more singular. This is true for deformations of Kleinian singularities.

▶ If $\lambda$ is quasi-dominant, $\mathcal{O}^\lambda(\widetilde{Q})$ is commutative, and $\mu = (\lambda_0 + 1, \lambda_1, \ldots, \lambda_n)$, then we think of $\mathcal{O}^\mu(\widetilde{Q})$ as a noncommutative analogue of $\mathcal{O}^\lambda(\widetilde{Q})$.

▶ They have the same singularity categories.

▶ If $\lambda = 0$ and $\mu$ is as above, then there is a noncommutative version of the geometric McKay correspondence.
Crawley-Boevey–Holland’s original paper introduced the deformations differently. Here’s what they actually did:

Let $G \leq \text{SL}(2, \mathbb{C})$ with associated extended Dynkin graph $\tilde{Q}$ and let $S = \mathbb{C}[u, v] \# G$.

Crawley-Boevey–Holland showed that one can deform $S$ to get an algebra $S^\lambda$ and that

1. $S^\lambda \sim \Pi^\lambda(\tilde{Q})$; and
2. if $e = \frac{1}{|G|} \sum_{g \in G} g$, then $eS^\lambda e \cong \mathcal{O}^\lambda(\tilde{Q})$.

Can we replace $\mathbb{C}[u, v]$ and $G$ with sensible alternatives and get similar results?
A noncommutative generalisation of CBH’s work

- Chan, Kirkman, Walton & Zhang recently classified all pairs \((A, H)\) where:
  - \(A\) is an AS-regular algebra of global dimension 2; and
  - \(H\) is a semisimple Hopf algebra acting inner faithfully on \(A\) with trivial homological determinant.

- These actions are like the actions of finite subgroups of \(\text{SL}(n, \mathbb{C})\) on \(\mathbb{C}[x_1, \ldots, x_n]\).

- CKWZ have shown that analogues of results in the Auslander-McKay correspondence for finite subgroups of \(\text{SL}(2, \mathbb{C})\) hold for the pairs \((A, H)\).

- I’ll restrict attention to the case where \(H = \mathbb{C}G\) for some group \(G\). How much of CBH’s work generalises?
The pairs \((A, G)\)

<table>
<thead>
<tr>
<th>Case</th>
<th>(A)</th>
<th>(G)</th>
<th>(\tilde{Q})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>(\mathbb{C}[u, v])</td>
<td>(G \leq \text{SL}_2(\mathbb{C}))</td>
<td>(\tilde{A}-\tilde{D}-\tilde{E})</td>
</tr>
<tr>
<td>(i)</td>
<td>(\mathbb{C}_q[u, v])</td>
<td>(C_{n+1})</td>
<td>(\tilde{A}_n)</td>
</tr>
<tr>
<td>(ii)</td>
<td>(\mathbb{C}_1[u, v])</td>
<td>(C_2)</td>
<td>(\tilde{L}_1)</td>
</tr>
<tr>
<td>(iii)</td>
<td>(\mathbb{C}_1[u, v])</td>
<td>(D_n)</td>
<td>(\begin{cases} \tilde{D}<em>{\frac{n+4}{2}} &amp; \text{n even} \ \tilde{DL}</em>{\frac{n+1}{2}} &amp; \text{n odd} \end{cases})</td>
</tr>
<tr>
<td>(iv)</td>
<td>(\mathbb{C}_J[u, v])</td>
<td>(C_2)</td>
<td>(\tilde{A}_1)</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\tilde{L}_1 & \quad \begin{tikzpicture}
\draw (0,0) circle (0.5cm);
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,-1);
\end{tikzpicture} \\
\tilde{DL}_n & \quad \begin{tikzpicture}
\draw (0,0) circle (0.5cm);
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,-1);
\draw (1,0) -- (2,0);
\draw (1,0) -- (1,-1);
\draw (2,0) -- (3,0);
\draw (2,0) -- (2,-1);
\draw (3,0) -- (4,0);
\draw (3,0) -- (3,-1);
\draw (4,0) -- (5,0);
\draw (4,0) -- (4,-1);
\draw (5,0) -- (6,0);
\draw (5,0) -- (5,-1);
\end{tikzpicture}
\end{align*}
\]
Deformations of $A \# G$ and $A^G$

- Fact: $e(A \# G)e \cong A^G$.
- One can deform the algebras $A \# G$ and $A^G$ in the same way as CBH did to get algebras $(A \# G)^\lambda$ and $e(A \# G)^\lambda e$, where $\lambda \in \mathbb{C} \tilde{Q}_0$.
- These deformations have nice properties:

**Proposition (C.)**

- $(A \# G)^\lambda$ is a prime, noetherian, finitely generated $\mathbb{C}$-algebra. It is Auslander-regular of global dimension $\leq 2$, and Cohen-Macaulay of GK dimension 2.
- $e(A \# G)^\lambda e$ is a finitely generated $\mathbb{C}$-algebra which is a noetherian domain. It is Auslander-Gorenstein, and Cohen-Macaulay of GK dimension 2.
Deformed “preprojective algebras”

- Fix $q \in \mathbb{C}^\times$. Define the quantum deformed preprojective algebra $\Pi_q^\lambda(\widetilde{A}_n)$ as the path algebra with relations

  $\alpha_i \alpha_i - q \alpha_{i-1} \alpha_{i-1} = \lambda_i e_i.$

- Define $\Delta^\lambda$ as the path algebra with relations

  $\alpha_0 \alpha_0 - \alpha_1 \alpha_1 - \alpha_0 \alpha_1 = \lambda_0 e_0$
  $\alpha_1 \alpha_1 - \alpha_0 \alpha_0 - \alpha_1 \alpha_0 = \lambda_1 e_1.$
Morita equivalences and isomorphisms between deformations

Let $G \leq \text{SL}(2, \mathbb{C})$ with associated extended Dynkin graph $\tilde{Q}$. Then Crawley-Boevey–Holland’s results can be written as

- $(\mathbb{C}[u, v] \# G)^\lambda \sim \Pi^\lambda(\tilde{Q})$; and
- $e(\mathbb{C}[u, v] \# G)^\lambda e \cong e_0 \Pi^\lambda(\tilde{Q})e_0$.

These results generalise to our new setting:

**Theorem (C., 2016)**

**Case (i):** $(\mathbb{C}_q[u, v] \# C_{n+1})^\lambda \sim \Pi_q^\lambda(\tilde{A}_n)$ and $e(\mathbb{C}_q[u, v] \# C_{n+1})^\lambda e \cong e_0 \Pi_q^\lambda(\tilde{A}_n)e_0$.

**Cases (ii)-(iii):** $(A \# G)^\lambda \sim \Pi^\lambda(\tilde{Q})$ and $e(A \# G)^\lambda e \cong e_0 \Pi^\lambda(\tilde{Q})e_0$.

**Case (iv):** $(\mathbb{C}_J[u, v] \# C_2)^\lambda \sim \Delta^\lambda$ and $e(\mathbb{C}_J[u, v] \# C_2)^\lambda e \cong e_0 \Delta^\lambda e_0$.
Auslander’s Theorem for the deformations

We have the following well-known theorem:

Auslander’s Theorem (1962)

Let \( G \leq \text{GL}(n, \mathbb{C}) \) be a small group acting on \( S := \mathbb{C}[x_1, \ldots, x_n] \). Then
\[
\text{End}_{S^G}(S) \cong S^\#G.
\]

Chan-Kirkman-Walton-Zhang recently proved the following:

Theorem (Chan-Kirkman-Walton-Zhang, 2016)

Let \((A, G)\) be a pair from the earlier table. Then
\[
\text{End}_{A^G}(A) \cong A^\#G.
\]

A slightly stronger result can be proved using different techniques:

Theorem (C., 2017)

The deformations \((A \# G)^\lambda\) are maximal orders, and
\[
\text{End}_{e(A \# G)^\lambda_e}(A) \cong (A \# G)^\lambda.
\]
Future questions

- What do the singularity categories of the deformations $e(A \# G)^\lambda e$ look like?
  - When $\lambda = 0$ (so $e(A \# G)^\lambda e \cong A^G$), I can answer this.

- How do the global dimensions of $(A \# G)^\lambda$ and $e(A \# G)^\lambda e$ vary with $\lambda$?

- How does the number of finite dimensional simple modules over $(A \# G)^\lambda$ and $e(A \# G)^\lambda e$ vary with $\lambda$?

- Is $(A \# G)^\lambda$ ever Morita equivalent to $e(A \# G)^\lambda e$?