Azumaya loci and discriminant ideals

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Aim and Plan

1. AIM: To explain and illustrate some old and some new results on the representation theory of algebras satisfying a polynomial identity.

2. PLAN:
    - Objects of study
    - Classical PI theory
    - Traces and discriminant ideals
    - Main result
    - Main result - idea of proof
    - Final comments

Objects of study

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$$R \quad \subseteq \quad Q(R) \quad = \quad R \otimes_{Z(R)} Q(Z(R)),$$

with

$$\dim_{Q(Z(R))} Q(R) = n^2.$$

The integer n is the PI-degree of R.

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and, for all simple R-modules V,

$$\dim_k(V) = 1.$$

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Let Z be a commutative affine domain, let m be a positive integer, and let $R = M_m(Z)$, $m \times m$ matrices over $Z = Z(R)$.

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$$\{(Isom.\ classes\ of)\ simple\ R - modules\} \approx \mathrm{Maxspec}(Z),$$

and

$$\dim_k(V) = m.$$

Objects of study (C)

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Let $T = M_2(k[X, Y])$, and let R be the subalgebra

$$R := \begin{pmatrix} k[X, Y] & \langle X, Y \rangle \\ k[X, Y] & k[X, Y] \end{pmatrix}$$

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So $\exists$ unique simple R-module V with $V\mathfrak{m}_{a,b} = 0$, and

$$\dim_k(V) = 2.$$

Objects of study (C), continued.

Example

But there are 2 maximal ideals of R containing $\mathfrak{m}_{0,0}$, namely

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We'll see that this pattern is typical.....

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2. *For a maximal ideal* $\mathfrak{m}$ *of* $Z(R)$, *there exist* t *simple R-modules V with* $V\mathfrak{m} = 0$, *where* $1 \leq t \leq n$.
3. *Define*

$$\mathcal{A}(R) := \{\mathfrak{m} \in \mathrm{Maxspec}(Z) : \exists V \text{ simple}, V\mathfrak{m} = 0, \dim_k V = n\}.$$

Then $\mathcal{A}(R)$ *is a non-empty open (hence dense) subset of* $\mathrm{Maxspec}(Z(R))$.

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(2) Describe the non-Azumaya simple modules.

We'll focus on (1) in the rest of the talk.

Traces and discriminant ideals

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Definition

Assume R satisfies (H). A trace map $\mathrm{tr} : R \longrightarrow Z(R)$ is a map which

1. is $Z(R)$-linear;
2. is non-zero;
3. satisfies the trace property, $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in R$.

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Define $\mathrm{tr}_{\mathrm{red}}$ to be the composition

$$R \hookrightarrow R \otimes_{Z(R)} Q(Z(R)) \hookrightarrow R \otimes_{Z(R)} F \cong M_n(F) \xrightarrow{\mathrm{tr}} F,$$

where tr denotes the usual matrix trace.

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$$MD_m(R, \mathrm{tr}) := \langle \det[\mathrm{tr}(y_i y_j')] : (y_1, \dots y_m), (y_1', \dots, y_m') \in R^m \rangle,$$

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Suppose R satisfies (H), with $R = \sum_{i=1}^{t} Z(R) b_i$. *Then*

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3. If $m > n^2$, then $MD_m(R, \mathrm{tr}) = \{0\}$.
4. If R is $Z(R)$-free on basis $\{b_1, \ldots, b_{n^2}\}$, then $MD_{n^2}(R, \mathrm{tr}) = \langle \det[\mathrm{tr}(b_i b_j)] \rangle$, a principal ideal of $Z(R)$.

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5. If
$$R := \begin{pmatrix} k[X,Y] & \langle X,Y \rangle \\ k[X,Y] & k[X,Y] \end{pmatrix},$$
then $MD_4(R, \mathrm{tr}_{\mathrm{red}}) = \langle X, Y \rangle$.

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The last example is very suggestive....

Main result

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$$\mathcal{V}(MD_{n^2}(R, \mathrm{tr}_{\mathrm{red}})) = \mathrm{Maxspec}(Z(R)) \setminus \mathcal{A}(R).$$

Main result - idea of proof

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Definition

1. Let R satisfy (H). A trace map $\mathrm{tr}\colon R \to Z(R)$ is representation theoretic if for all $\mathfrak{m} \in \mathrm{Maxspec}(Z(R))$ there exists a non-trivial finite dimensional $R/\mathfrak{m}R$-module $W_\mathfrak{m}$ and a scalar $s_\mathfrak{m} \in k^*$ (both depending on $\mathfrak{m}$) such that the following diagram commutes:

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2. Say tr is almost rep. theoretic if the above holds with $s_{\mathfrak{m}} \in k$, but $s_{\mathfrak{m}} \in k^*$ whenever $R/\mathfrak{m}R$ is simple.

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Call the form nondegenerate if $\langle u, B \rangle = 0$ only for $u = 0$.

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Call the form nondegenerate if $\langle u, B \rangle = 0$ only for $u = 0$.
The kernel $\{u \in B : \langle u, B \rangle = 0\}$ of the form is an ideal of B;

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Let R satisfy (H), with $Z(R)$ normal.

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In this case, for basis $\{b_1, \ldots, b_t\}$ of B, $\det[\operatorname{tr}(b_i b_j)]_{t \times t} \neq 0$.

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This proves the Main Theorem.

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