Was Pythagoras Chinese?

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Introduction

This article presents two self-contained proofs of the Pythagorean Theorem that are strictly geometric, involving neither measurements nor numbers. The first might have been discovered by Pythagoras in the sixth century BC. The second is due to Liu Hui from about 300 AD. The two proofs show how mathematicians in two ancient civilizations—one in the West (ancient Greece) and the other in the East (ancient China)—deduced a result about right triangles from strictly geometric arguments. We also briefly contrast the geometric approaches with an arithmetic method employed by mathematicians from a third great ancient civilization—the Babylonians. The question posed in the title of this article is borrowed freely from a book by Frank J. Swetz and T. I. Kao [7]. Our purpose here is to show how the radically different civilizations in China and Greece regarded right triangles in a remarkably similar way.

The material in this article is appropriate for students taking geometry for the first time in high school (or perhaps earlier); we provide suggestions for using cut-outs to help visualize the process. The only notion that is assumed is the concept of *congruence*, yet even here it is used in the intuitive sense of placing one figure precisely on top of another. The greatest benefit for beginning students might be an understanding of the nature of *mathematical proof*, because the historical approach adopted here illustrates a type of intuitive argument (based on obvious properties of figures) that preceded formal chains of reasoning that characterize deduction. To better appreciate the two geometric proofs, we describe a tactile approach using congruent triangles (formed by cutting two pieces of a rectangular sheet of paper along a diagonal) and various squares.

Figure 1 shows the familiar 3-4-5 right triangle. The Pythagorean Theorem asserts that the square *on* 5 is equal to the square *on* 3 plus the square *on* 4. Sometimes today we think of squaring as an arithmetic operation, for instance, when writing the Pythagorean Theorem as the square *of* 5 is equal to the square *of* 3 plus the square *of* 4. The distinction between the italicized words *on* and *of* is important. The former reflects a fundamentally geometric idea expressing a relationship among squares constructed on the sides of a right triangle. The latter is an arithmetic expression that harkens back 4000 years to Babylon, whose mathematically adroit scientists uncovered numerous arithmetic properties of right triangles. The most important one determined those triples of integers that could represent the lengths of the legs of right triangles. However, we are not concerned with such arithmetical relationships here.

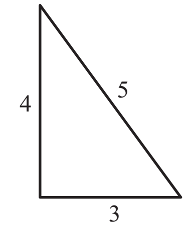

Figure 1

Right triangles in Ancient Greece

Jacob Bronowski (1908–1974) was a scientist of encyclopedic learning who found both the arts and the sciences interesting and accessible. He also had an innate ability to explain complex principles to a general audience. So in 1969, when the BBC sought a scientific counterpart to their highly successful series on Western art called *Civilisation*, they enlisted Bronowski as host and writer. The result was a 13-part series called *The Ascent of Man* that presented the history of science within a cultural history of mankind. The series was an overwhelming success when it appeared on PBS stations in the U.S. in 1973. *Ascent* turned out to be Bronowski's last project; it was completed shortly before his death in New York from a heart attack at age 66. (An informative web site with a short biography of Bronowski, containing extracts from his writings and personal reminiscences from those who knew him, is www.drbronowski.com/.)

The book *Ascent of Man* [1] was published when the show was televised. Chapter 5—correspondingly Episode 5 in the TV series—is devoted to the basic philosophy and lifestyle of the Pythagoreans. Titled "The Music of the Spheres," it emphasizes the Pythagorean credo that mathematical relationships form the essence of all phenomena, and that such relationships alone should be used to express seemingly unrelated events. This particular episode describes the manner in which the Pythagoreans reduced both music and the motions of planets to simple relationships among numbers. It was not until the time of Galileo in the early seventeenth century that scientists again began to look for the essence of things in number. Today this philosophy applies to almost all physical sciences and many social sciences as well. Yet, as we will see, the Pythagoreans could also demonstrate results in a totally geometric manner. The practice of justifying statements on geometric reasoning alone continued through the end of the seventeenth century, when Isaac Newton employed strictly geometric proofs of all results in his historic discovery of calculus.

This section examines a possible proof of the Pythagorean Theorem. By "proof" we mean a convincing demonstration of the correctness of a result. The proof here is a reconstruction due to Jacob Bronowski, who based it on the geology of the region in southern Italy where the Pythagoreans settled. The method of proof lies somewhere between the experimental approach of the ancient Egyptians (from about 1650 BC) and the purely deductive scheme adopted by Euclid in ancient Greece (about 300 BC). We highly recommend that the reader view "The Music of the Spheres" to see how natural the proof of the Pythagorean Theorem might have appeared in Croton, the Italian seaport ultimately inhabited by the merry band of philosophers called the Pythagoreans. Nature presents us with many shapes, like the tiles that dot the Mediterranean coastline. Such tiles often appear in the shape of triangles. (See Figure 67 in [1].)

We describe how Pythagoras's line of reasoning might have been based on tiles to justify the result now named after him. Begin with the right triangle shown in Figure 2. For classroom demonstrations we suggest that the teacher bisect two 8½×11 pieces of paper to form four triangles. We recommend that students bisect 3×5 cards.

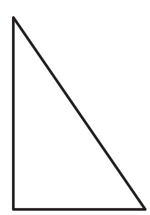

Figure 2

Take a second triangle (a congruent copy of the one in Figure 2), rotate it 90° clockwise, and place it as shown in Figure 3. An important part of Figure 3 is the dotted line segment on the longer side of the second (bottom) triangle. This segment is the difference between the longer side and the shorter side of the triangle. Equivalently, a line segment that is composed of the shorter side plus the dotted line segment is congruent to the longer side.

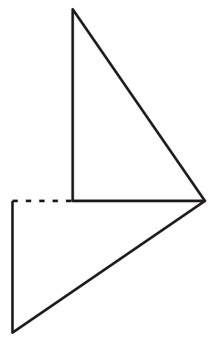

Figure 3

Next, take a third (congruent) triangle and rotate it 180° clockwise from the initial position. Finally, take a fourth triangle and rotate it 270° clockwise from the initial position. Place the four triangles as shown in Figure 4; we number them in their order of construction. If you were to take the initial triangle and rotate it 360° clockwise it would end up where it began. That is the crucial property of a 90°-rotation: after four repetitions, any figure, not just a triangle, will end up exactly where it began.

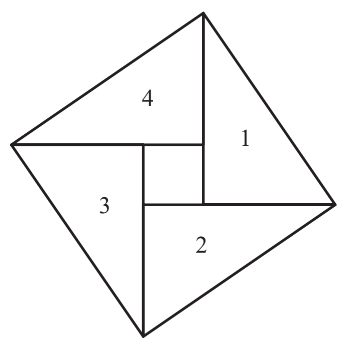

Figure 4

The diagram in Figure 4 is a square (we will call it the *outer square*) composed of four congruent triangles (think of them as four identical tiles) and the *inner square*. Notice that each side of the outer square is the hypotenuse of the right triangle, so its area is equal to the area of a square on the hypotenuse of the right triangle. At this point it is advisable to construct a replica of the inner square; otherwise it might get lost. Each side of this square is the difference between the longer and shorter legs of the triangle; it can be constructed using measurements, if necessary.

The outer square is composed of the four triangles plus the inner square. We aim to rearrange these five tiles—the inner square tile and the four triangular tiles—so they form two square tiles. Physically it is rather easy to slide a tile from one location to another. Begin by sliding tile 1 under tile 3 to form a rectangle; then slide tile 4 under tile 2 to form another rectangle. This results in the L-shaped figure (called a *gnomon*) shown in Figure 5, which has the same area as the outer square in Figure 4 since it is composed of the same tiles.

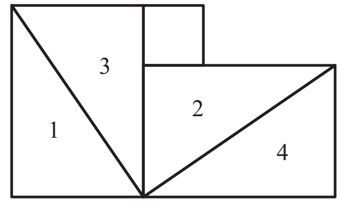

Figure 5

The final step is to draw the vertical dotted line shown in Figure 6 to help visualize how the L-shaped figure can be partitioned into two squares. Look at the figure situated to the right of the dotted line. Each vertical side is the shorter side of the right triangle. Each horizontal side is the difference between the longer side of the right triangle and the side of the inner square, so each one is the shorter side of the triangle. Also, all angles at the four vertices are right, so the figure is a square on the shorter side of the right triangle. Similarly the figure to the left of the dotted line is a square on the longer side of the right triangle. Consequently, the diagram in Figure 6 has been partitioned into two squares, so its area is the sum of the area of the square on the shorter side plus the area of the square on the longer side. Since Figure 6 was formed from Figure 4 by moving tiles whose areas remain the same, the areas of the diagrams in Figure 4 and Figure 6 are the same. Therefore the area of the square on the hypotenuse of the original triangle (Figure 4) is equal to the sum of the area of the square on the shorter side plus the area of the square on the longer side (Figure 6).

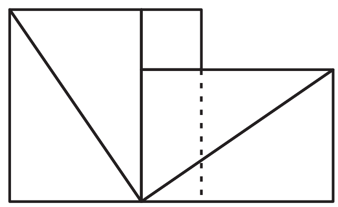

Figure 6

This completes the proof of the Pythagorean Theorem—that's all there is to it. Because there is no extant documentation from the sixth century BC when Pythagoras lived, we have no idea how his followers proved his eponymous result. The proof given here was suggested by Jacob Bronowski 30 years ago.

Bronowski's tactile reconstruction of Pythagoras's proof using tiles does not involve any calculations. No arithmetic was performed—the method is entirely geometric. This approach shows how initially the Pythagorean Theorem dealt with geometric squares and not with arithmetic operations. Next we examine a similar approach that was adopted in an entirely different part of the world—China.

Right triangles in Ancient China

Euclid's famous work on Greek geometry, the *Elements*, was composed about 300 BC, some 200 years after the death of Pythagoras. By contrast, and based on written records, Chinese mathematics is not as old as its Greek counterpart. By the time of the Han dynasty (208 BC – 8 AD), however, there were several Chinese works that reflected a well-developed subject area. One of the major books from this period, *Jiuzhang suanshu*, has been described as the "classic of classics." The title can be translated into English as *Nine Chapters on the Art of Mathematics*, which we shorten to *Nine Chapters*. Just as the *Elements* reflected developments in mathematics since the time of Pythagoras, *Nine Chapters* is a compilation of works written before it. Also like the *Elements*, subsequent commentaries on *Nine Chapters* provided a major outlet for further advances in the field.

Here we examine an impressive, convincing, colorful proof of the Pythagorean Theorem taken from a commentary on *Nine Chapters* by the Chinese mathematician Liu Hui about 300 AD. We will see that Liu Hui's proof is very similar to Bronowski's reconstruction of the Pythagorean proof. The close connection spawned the title of this article. We begin with the same right triangle as Figure 2 from the Pythagorean approach. Here the first step is to construct an inner square on the shorter side in the manner shown in Figure 7. The Chinese colored the interior of this square red; we use dark gray.

Next, construct an exterior square on the longer side of the original right triangle as shown in Figure 8. The Chinese colored the interior of this square blue; we use light gray.

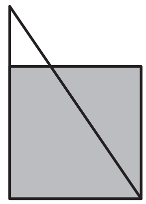

Figure 7

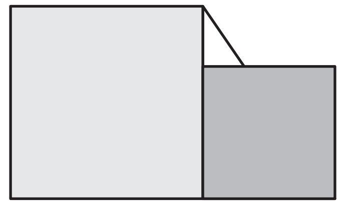

Figure 8

Now construct a congruent copy of the original right triangle. Situate it with the same orientation so that its longer side overlaps the lower-left corner of the light gray square by a triangle congruent to the unshaded triangle in Figure 8. This means that triangles A and B in Figure 9 are congruent, which enables us to transfer the light gray color from A to B. Chinese mathematicians referred to the process of transferring colors as "out-in mutual patching."

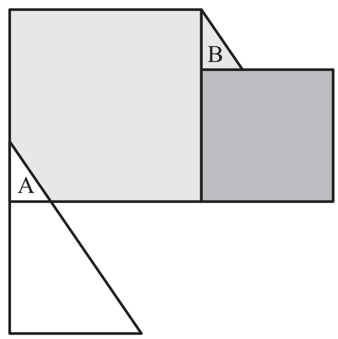

Figure 9

Next, make a copy C of the original right triangle, rotate it by 90° clockwise, and place it into the position shown in Figure 10. Then remove the light gray color from another copy D of the triangle onto triangle C; the transfer of this color is also shown in Figure 10.

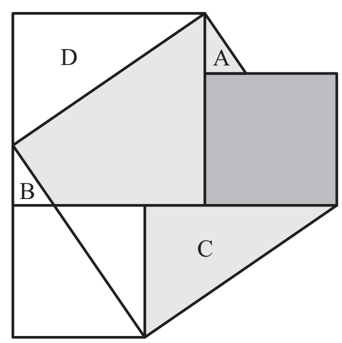

Figure 10

So far no dark gray color has been transferred. Can you anticipate the final step involving such a transfer? Look at triangles E and F in Figure 11. They are congruent right triangles with one leg equal to the shorter side of the original right triangle, and the other leg equal to the difference between the shorter side and a leg of the congruent triangles A and B. Therefore the final step is to transfer the dark gray color from triangle E to triangle F.

The shaded region in Figure 11 (consisting of both light gray and dark gray regions) forms a square on the hypotenuse of the original right triangle. The light gray part of Figure 11 initially formed a square on the longer side of the triangle, while the dark gray part formed a square on the shorter side. Because the area of the region that is shaded equals the area of the light gray region plus the area of the dark gray region, it follows that the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the shorter side and the longer side.

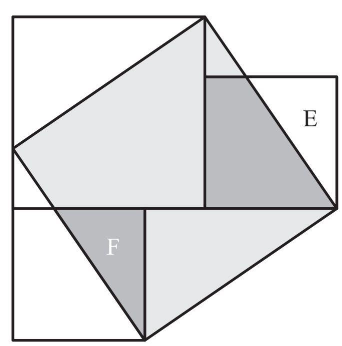

Figure 11

This completes the colorful Chinese proof of the Pythagorean Theorem. It is important to point out that Liu Hui's proof, like Pythagoras's, did not use calculations; both are entirely geometric, with no arithmetic computations performed.

Was Pythagoras Chinese?

It is especially difficult to answer questions about Chinese history during the time that *Jiuzhang suanshu* was written because in 213 BC the emperor Shih Huang-ti ordered all books to be burned. (See [3] and [5] for excellent accounts of Chinese mathematics.) The history of the West makes the question of priority of results and independence of proofs troublesome for the Greeks too. (The book by Heath [4] remains an excellent source for this history.) However, the proofs of the Pythagorean Theorem by Pythagoras (as reconstructed) and by Liu Hui are typical of the types of justification that convinced others of the correctness of geometrical statements about 2500 years ago. The nature of the Greek and Chinese proofs is almost identical. (The textbook [8] provides a similar development.) This raises three questions:

1. Is it possible that one civilization borrowed from the other?
2. Might Pythagoras have preceded Marco Polo in his travels to the Orient?
3. Might an unknown Chinese businessman, schooled in mathematics, have traveled to the Mediterranean area and come in contact with Pythagoreans?

Before asking another question, we hasten to add that there is not a shred of evidence to suggest answers to these three questions. Lacking historical evidence, we can only conclude that Greek and Chinese mathematicians proceeded independently. Now we pose a question for anyone familiar with Euclid's proof of the Pythagorean Theorem (Book 1, Proposition 47).

4. Which proof is more convincing—the one by Euclid or those due to Pythagoras and Liu Hui?

We expect most people familiar with Euclid's proof to answer Question 4 with "those due to Pythagoras and Liu Hui" because they are immediately convincing. This suggests that school districts might improve geometric instruction if a first approach is along historical lines, perhaps enhanced by an excursion into the three-dimensional analog. (For further details, please see "When is a Square Square and a Cube Cubical" by Amy Shell-Gellasch in this volume.) However, an historical approach must contain a vital caveat for anyone desiring to return to the glory days of these ancient civilizations: depending entirely on figures and drawing conclusions from experience can lead to incorrect results. Deduction arose as a response to this shortcoming, and it has served as the mathematical standard since the time of Euclid. (Section 3-6 in [2] discusses how the method of deduction arose, with ties to the ancient Egyptian culture.) Even deduction took a drastic hit by the early 1930s when two logicians, Emil Post and Kurt Gödel, proved independently that in any system like the one needed to demonstrate the Pythagorean Theorem, there will be some statements that can be neither proved nor disproved.

We end this presentation with a question for beginning students.

5. Which proof do you find more convincing—the Chinese or the Greek?

Directions for the instructor

The author has found success with students repeating the Greek proof of the Pythagorean Theorem. However, replicating the Chinese method has been more elusive, and generally results in a demonstration using transparencies. Nonetheless, in this final section we provide instructions for students to perform both proofs.

For the Greek approach, each student will need four congruent right triangles, and a square whose side is the difference between the longer side and the shorter side of the triangle. We instruct each student to cut two 3×5 cards in half to form the four triangles, and then to construct the inner square from a third 3×5 card. The resulting "tiles" are small enough to be manipulated on the student's desk top, yet big enough to be convincing.

Initially, all students should place the four triangles and square as shown in Figure 4. It is important for students to see that each side of the outer square is the hypotenuse of the original triangle. Ask, "How can these five tiles be rearranged to form two squares?" The object is for the students to slide the tiles to the positions shown in Figure 5, and to realize how two squares result. This realization can be confirmed by constructing squares of appropriate size and placing them on top of the gnomon in Figure 6.

Having students reconstruct the Chinese method is not as easy. We think it is best to use graph paper and color appropriate regions, an approach that resonates with the Chinese method. However, we have constructed a method using squares and triangles that can be cut from a sheet of paper with inches as the basic unit. First cut a 4×4 square (denoted S4) and use a red pencil to color it. Next, cut a 3×3 square (denoted S3) and use a regular lead pencil to shade it. Then cut six 3-4-5 right triangles, labeled T0, T1, T2, T3, T4, and T5, with T3 colored red. The proof follows from these five steps.

1. Place S3 atop T0 and S4 to the left of T0 (Figure A).
2. Place T1 at the top left corner of S4 and T2 directly below it (Figure B).
3. Place the red T3 under S3 as shown in Figure C. Then transfer the red from triangle A to triangle B by coloring B.
4. Place T4 atop S3 and to the right of T0. Color the triangular part of T4 that covers S3 with the lead pencil (Figure D).
5. Move T4 into the position in Figure E so its colored region fills the gap, and then place T5 where T4 had been to complete the transfer of color.

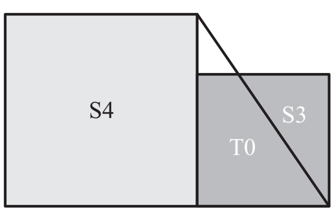

Figure A

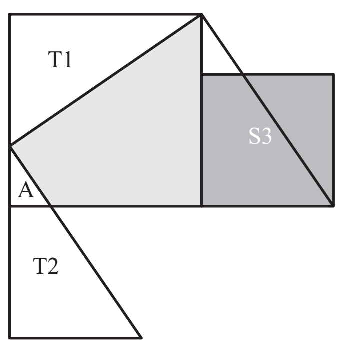

Figure B

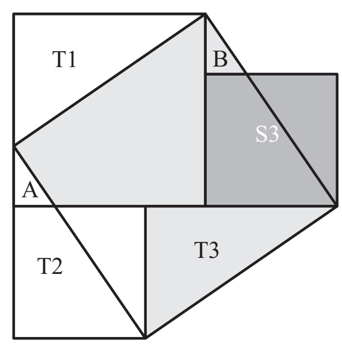

Figure C

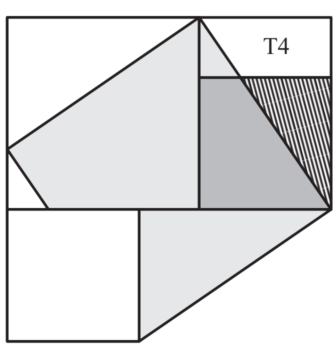

Figure D

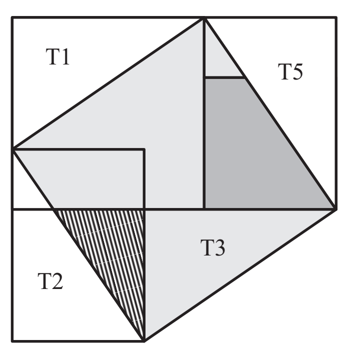

Figure E

Beyond Pythagoras and Liu Hui

Euclid's book, generally referred to as the *Elements*, consists of thirteen chapters that were then called "books." Major results were called "propositions." Book I consists of 48 propositions, and the penultimate one is the Pythagorean Theorem, labeled merely Proposition I-47. It reads:

> *In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.*

Notice that Euclid uses the expression "square on the side" instead of "square of the side." This subtle difference reinforces the geometric nature of the Pythagorean Theorem, as opposed to the arithmetic character suggested by the alternate wording. Yet Euclid went beyond Pythagoras. Proposition 31 in Book VI reads:

> *In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.*

This result is an impressive generalization of the Pythagorean Theorem. It says, for instance, that the area of an equilateral triangle placed on the hypotenuse of a right triangle is equal to the sum of the areas of equilateral triangles placed on the two legs. That extension would take care of three-sided figures on the legs. The Pythagorean Theorem itself deals with four-sided figures.

Proposition VI-31 goes beyond quadrilaterals, and includes regular pentagons placed on the three sides of a right triangle, regular hexagons, etc. In short, Euclid's generalization gives assurance that the area of a regular polygon on the hypotenuse is equal to the sum of the areas of regular polygons on the legs. And one need not restrict attention to polygons—Proposition VI-31 includes such figures as semicircles as well. However, an examination of additional extensions would take us too far afield. For more details, the interested reader is referred to a recent, charming paper [6].

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