PROBLEM SET 12

Theoretical Linear Algebra 3051 **Daniel B. Szyld**

Due Tuesday April 20

Let V be an inner product space over the field $\mathbb{F}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be complementary subspaces of V, i.e., $V = \mathcal{X} \oplus \mathcal{Y}$, with $\dim \mathcal{X} > 0$ and $\dim \mathcal{Y} > 0$. Additionally assume that $\mathcal{Y} \neq \mathcal{X}^{\perp}$.

1. Show that $V = \mathcal{Y}^{\perp} \oplus \mathcal{X}^{\perp}$.

2. Show that $\dim \mathcal{X} = \dim \mathcal{Y}^{\perp}$.

3. Either prove or give a counter-example:
Let $x \in \mathcal{X}$, $y \in \mathcal{Y}$. There exists $u \in \mathcal{Y}^{\perp}$, $v \in \mathcal{X}^{\perp}$ such that $\|u\| = \|x\|$, $\|v\| = \|y\|$, and $\langle u, v\rangle = \langle x, y\rangle$.

4. Let P be the projection such that range $P = \mathcal{X}$ and null $P = \mathcal{Y}$.
Show that P^* is the projection with range $P^* = \mathcal{Y}^{\perp}$ and null $P^* = \mathcal{X}^{\perp}$.

5. For such a projection P (as in exercise 4.) show that $\|P\| = \|I - P\|$.