Homework 8 Math 8052, Spring 2009

The following problems illustrate parts of the proof of Runge's Theorem. We let $K = \{z : |z| \leq 1/2\}$, $a_0, a_1 \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$, *and* $b = (a_0 + a_1)/2$.

Part I

1. Find the optimal condition on a_1/a_0 that guarantees that $|b - a_0| < 1/2$.

Assume until further notice that $|b - a_0| < 1/2$ *(in particular, for such* b *we have* $b \notin K$*). Then the function* $1/(z - b)$ *is uniformly approximable on* K *by rational functions which are polynomials in* $f = 1/(z - a_0)$ *with pole at* a_0*. This is made explicit in the following problem*

2. Show that

$$\sup_{z \in K} \left| \frac{1}{z - b} - \frac{1}{z - a_0} \sum_{k=0}^{n} \frac{(b - a_0)^k}{(z - a_0)^k} \right| \leq 2 \frac{(2|b - a_0|)^{n+1}}{1 - 2|b - a_0|}.$$

(Observe that the right hand side tends to 0 as n teds to ∞.)

3. Define

$$R_n(z) = \frac{1}{z - a_0} \sum_{k=0}^{n} \frac{(b - a_0)^k}{(z - a_0)^k}.$$

Show that

$$\sup_{z \in K} |R_n(z)| \leq \frac{2}{1 - 2|b - a_0|}.$$

4. With the notation of the previous problem, estimate

$$\sup_{z \in K} \left| \frac{1}{(z - b)^m} - R_n(z)^m \right|.$$

Your estimate should be an expression in n and $|b - a_0|$ that tends to 0 as n tends to ∞. Hint: Use the binomial theorem.

5. Now find the optimal condition on a_1/a_0 that guarantees that in addition to $|b - a_0| < 1/2$, also $|a_1 - b| < \operatorname{dist}(b, K)$.

Assume henceforth that $|b - a_0| < 1/2$ *and* $|a_1 - b| < \operatorname{dist}(b, K)$*. Note this time that then the function* $1/(z - a_1)$ *is uniformly approximable on* K *by a rational functions which are polynomials in* $1/(z - b)$*.*

6. Let

$$Q_n(z) = \frac{1}{z - b} \sum_{k=0}^{n} \frac{(a_1 - b)^k}{(z - b)^k}$$

Give (prove) an estimate for

$$\sup_{z \in K} \left| \frac{1}{z - a_1} - Q_n(z) \right| \tag{$\dagger$}$$

similar to the estimate in Problem 2.

7. Use problem 4 to also give an estimate for

$$\sup_{z \in K} \left| Q_n(z) - \sum_{k=0}^{n} (a_1 - b)^k R_m(z)^{k+1} \right|.$$

Assume m is as large as is convenient in your argument.

8. Use your estimate in Problem 4 to give an estimate for

$$\sup_{z\in K}\left|Q_n(z)^m-\left(\sum_{k=0}^{n}(a_1-b)^kR_n(z)^{k+1}\right)^m\right|$$

9. Use your estimate for (†) and an estimate similar to that of Problem 3 to find an estimate for

$$\sup_{z\in K}\left|\frac{1}{(z-a_1)^m}-Q_n(z)^m\right|.$$

Part II

10. Combine all the problems so far to conclude that if a is an *arbitrary* point in S^1, then for every $\varepsilon>0$ there is a rational function with pole only at 1 such that

$$\sup_{z\in K}\left|\frac{1}{z-a}-R(z)\right|<\varepsilon.$$

Let $G=\{z:|z|<1\}$ and $E=\{1\}$. The function $f(z)=z$ is holomorphic in G, of course, and by the Cauchy representation formula,

$$z=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2i\theta}}{e^{i\theta}-z}\,d\theta,\qquad z\in K. \tag{‡}$$

Let

$$F(z,\theta)=\frac{1}{2\pi}\frac{e^{2i\theta}}{e^{i\theta}-z}.$$

11. Let $C_0>0$ be such that

$$\sup_{z\in K,\ \theta\in\mathbb{R}}\left|\frac{\partial F}{\partial\theta}(z,\theta)\right|\le C_0$$

(such C_0 can be found explicitly). Use the formula

$$F(z,\theta)-F(z,\psi)=\int_0^1\frac{d}{dt}\big(F(z,\psi+t(\theta-\psi)\big)\,dt$$

to show that

$$|F(\theta,z)-F(\psi,z)|\le C_0|\theta-\psi|\quad\text{for } z\in K.$$

12. Show that

$$\sup_{z\in K}\left|z-\sum_{\ell=0}^{n-1}F\big(z,\frac{2\pi\ell}{n}\big)\frac{2\pi}{n}\right|\le\frac{(2\pi)^2C_0}{n}.$$

Hint: Write z as the integral (‡), then write the integral as

$$\sum_{\ell=0}^{n-1}\int_{\frac{2\pi\ell}{n}}^{\frac{2\pi(\ell+1)}{n}}F(z,\theta)\,d\theta,$$

and use the estimate in the previous problem to estimate the difference of interest.

The functions $z\mapsto F(z,2\pi\ell/n)$ are rational functions with pole only some $a_1\in S^1$. By the problems in the first part, these functions are uniformly approximable on K by a rational function with pole only at 1. Hence the same is true for z.