1. Let $G \subset \mathbb{C}$ be open and $a \in G$.

(1) Show that
$$\mathscr{I}_a = \{f \in \mathrm{Hol}(G) : f(a) = 0\}$$
is a maximal ideal in $\mathrm{Hol}(G)$. Hint: Let $\mathbf{1} : G \to \mathbb{C}$ denote the constant function with value 1. If $f \in \mathrm{Hol}(G)$, then $f = f(a)\mathbf{1} + (f - f(a)\mathbf{1})$. Note that $(f - f(a)\mathbf{1}) \in \mathscr{I}_a$.

(2) Give an example of an ideal $\mathscr{J} \subset \mathrm{Hol}(G)$, $\mathscr{J} \neq \mathscr{I}_a$, all whose elements vanish at a.

2. Let $f : G \to \mathbb{C}$ be meromorphic, not identically 0, let $\mathscr{P}$ be the set of poles of f and $\mathscr{Z}$ the set of zeros. Show that there are functions $r_{\mathscr{P}} : \mathscr{P} \to (0, \infty)$ and $r_{\mathscr{Z}} : \mathscr{Z} \to (0, \infty)$ such that the sets
$$\bigcup_{a \in \mathscr{P}} B(a, r_{\mathscr{P}}(a)), \quad \bigcup_{a \in \mathscr{Z}} B(a, r_{\mathscr{Z}}(a))$$
are disjoint and contained in G.

3. Let G be a proper open subset of $\mathbb{C}$.

(1) Show that for every $z \in G$ there is $w \in \mathbb{C}\backslash G$ such that $|z - w| = \mathrm{dist}(z, \mathbb{C}\backslash G)$.

(2) Assume now that G has the property that
$$\text{there is } R > 0 \text{ such that } \{z : |z| > R\} \subset G.$$
Let $\{z_n\}$ be a sequence in G without points of accumulation in G and such that $|z_n| \leq R$ for all R. For each n pick $w_n \in \mathbb{C}\backslash G$ such that $|z_n - w_n| = \mathrm{dist}(z_n, \mathbb{C}\backslash G)$. Show that $\lim_{n\to\infty} |z_n - w_n| = 0$.

4. Give an example of a proper open subset of $\mathbb{C}$ and a sequence $\{z_n\}$ in G without points of accumulation in G such that with whatever choice of $w_n \in \mathbb{C}\backslash G$, $|z_n - w_n|$ does not tend to 0.

5. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Show that there is $f \in \mathrm{Hol}(D)$ with the following property. For each $z_0 \in \partial D$ and each $r > 0$ there is no holomorphic function on $B(z_0, r)$ which coincides with f on $D \cap B(z_0, r)$.