Homework 9, due November 19 — Math 4041, Fall 2009

Setup: Ω is a bounded open set in $\mathbb{R}^2$ or $\mathbb{R}^3$ with C^1 boundary, $\vec{n}$ denotes the unit outer normal vector field, a, b are real-valued functions defined on $\partial\Omega$ such that $a(p)$ and $b(p)$ are not zero simultaneously for any $p \in \partial\Omega$, and

$$\mathcal{D} = \{u \in C^2(\overline{\Omega}) : u \text{ is complex-valued and } au + b\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega\}.$$

By dV we mean $dx\,dy$ or $dx\,dy\,dz$ depending on the dimension of Ω, and by dS we mean differential of length or area, again depending on the dimension.

1. Let $g \in C^1(\overline{\Omega})$ and let $\vec{F}$ be a vector field whose components are functions in $C^1(\Omega)$. Show

$$g \operatorname{div} \vec{F} = \operatorname{div}(g\vec{F}) - \vec{F} \cdot \operatorname{grad} g.$$

2. Use the previous problem and the divergence theorem to get

$$\int_\Omega \Delta f\, g\, dV = \int_{\partial\Omega} g\, \frac{\partial f}{\partial \vec{n}}\, dS - \int_\Omega \operatorname{grad} f \cdot \operatorname{grad} g\, dV, \quad f, g \in C^2(\Omega).$$

3. Use the previous problem to establish that

$$\int_\Omega \Delta u\, \overline{v}\, dV - \int_\Omega u\, \overline{\Delta v}\, dV = 0 \quad \text{if } u, v \in \mathcal{D}.$$

(Note that $\overline{\Delta v} = \Delta \overline{v}$.)

4. Use the previous problem to show that if $\lambda \in \mathbb{C}$ and if $u \in \mathcal{D}$ satisfies $\Delta u = -\lambda u$ and $u \neq 0$, then λ is a real number. (Hint: set $v = u$.)

5. Suppose the function b is strictly positive and $a \geq 0$. Show that if λ is as in the previous problem, then $\lambda \geq 0$.