1. Let

$$X(r,\theta) = r\cos\theta, \qquad Y(r,\theta) = r\sin\theta)$$

and let $F(r,\theta) = \big(X(r,\theta), Y(r,\theta)\big)$. Let u be a C^1 function defined in an open set Ω of $\mathbb{R}^2$ with $(0,0) \notin \Omega$. Define $v = u \circ F$. Use the chain rule to get explicit formulas for $\partial v/\partial r$ and $\partial v/\partial\theta$:

$$\begin{cases} \dfrac{\partial v}{\partial r}(r,\theta) = \alpha_{1,1}(r,\theta)\dfrac{\partial u}{\partial x}\big(F(r,\theta)\big) + \alpha_{1,2}(r,\theta)\dfrac{\partial u}{\partial y}\big(F(r,\theta)\big) \\ \dfrac{\partial v}{\partial \theta}(r,\theta) = \alpha_{2,1}(r,\theta)\dfrac{\partial u}{\partial x}\big(F(r,\theta)\big) + \alpha_{2,2}(r,\theta)\dfrac{\partial u}{\partial y}\big(F(r,\theta)\big) \end{cases}$$

Viewing this as a 2×2 system of equations, write $\dfrac{\partial u}{\partial x}\big(F(r,\theta)\big)$ and $\dfrac{\partial u}{\partial y}\big(F(r,\theta)\big)$ in terms of $\dfrac{\partial v}{\partial r}(r,\theta)$ and $\dfrac{\partial v}{\partial \theta}(r,\theta)$.

2. With the setup and notation of the previous problem, compute $\dfrac{\partial^2 u}{\partial y^2}\big(F(r,\theta)\big)$ in terms of $\dfrac{\partial v}{\partial r}(r,\theta)$ and $\dfrac{\partial v}{\partial \theta}(r,\theta)$.

3. Let b_0 and b_1 be constants with $b_0 \neq 0$. Determine what p and the coefficients u_k have to be (assuming $u_0 \neq 0$) in order for

$$u = x^p \sum_{k=0}^{\infty} u_k x^k$$

to be a solution of

$$xu'(x) - (b_0 + b_1 x)u(x) = 0 \quad \text{ in } x > 0.$$

Compare your series solution with the solution obtained by integrating $b_0/x + b_1$.