In the following problems we let a, $b > 0$ and $\mu_k = k\pi/a$, $k = 1, 2, \dots$.

1. Let

$$q_k(y) = -\frac{e^{\mu_k(y-b)} - e^{-\mu_k(y-b)}}{e^{\mu_k b} - e^{-\mu_k b}}, \qquad k \in \mathbb{N}.$$

Fix some such k and define

$$u(x, y) = q_k(y) \sin\Big(\frac{\pi k}{a} x\Big).$$

Verify that $\Delta u = 0$ and that

$$u(x, 0) = \sin\Big(\frac{\pi k}{a} x\Big),\ u(x, b) = 0 \text{ for } 0 < x < a,$$
$$u(0, y) = u(a, y) = 0 \text{ for } 0 < y < b.$$

2. For q_k as in the previous problem show:

a) For all δ with $0 < \delta < b/2$ there are C_0 and C_1 such that

$$\text{for all } k: \ |q_k'(y)| \le C_0 e^{-C_1 k} \text{ if } \delta < y < b - \delta$$

b) Same kind of estimate, but for $q_k''(y)$.

3. Let $a > 0$, let

$$u_0(x) = \begin{cases} x & \text{if } 0 \le x \le a/2 \\ 0 & \text{if } a/2 < x \le a. \end{cases}$$

Compute the integrals

$$c_k = \frac{2}{a} \int_0^a u_0(s) \sin\Big(\frac{\pi k}{a} s\Big)\, ds, \qquad k \in \mathbb{N}$$

and verify:

a) $\sum_{k=1}^{\infty} |c_k|$ does not converge;

b) $\sum_{k=1}^{\infty} |c_k|^2$ converges.

You may resort to quoting results from your calculus book.

4. Let a, $b > 0$ and let u_0 be the function of the previous problem. Find the solution of

$$\begin{cases} \Delta u = 0 \text{ in } 0 < x < a,\ 0 < y < b, \\ u(x, 0) = u_0(x, 0),\ u(x, b) = 0 \text{ for } 0 < x < a, \\ u(0, y) = u(a, y) = 0 \text{ for } 0 < y < b. \end{cases}$$