1. Show that

$$D_N(x) = \frac{\sin\big((N+1/2)x\big)}{\sin(x/2)},$$

initially defined for $x \notin 2\pi\mathbb{Z}$, has a limit at each $x_0 \in 2\pi\mathbb{Z}$. Show that D_N is 2π-periodic.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be C^2 functions. Let c be some positive number, and let $u(x,t) = f(x-ct) + g(x-ct)$. Show that u satisfies

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \text{ for } (t,x) \in \mathbb{R}^2.$$

3. Evaluate

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(y) e^{-iky}\, dy, \quad k \in \mathbb{Z},$$

where

$$u(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ x & \text{if } 0 \le x \le \pi. \end{cases}$$