Homework 10, due December 1 Math 4041, Fall 2009

The following problems will guide you in obtaining a formula for the Laplacian in $\mathbb{R}^3$ *in spherical coordinates. Define*

$$X(r,\theta,\phi) = r\cos\theta\sin\phi, \quad Y(r,\theta,\phi) = r\sin\theta\sin\phi, \quad Z(r,\theta,\phi) = r\cos\phi$$

and

$$F(r,\theta,\phi) = (X(r,\theta,\phi), Y(r,\theta,\phi), Z(r,\theta,\phi))$$

for $r \geq 0$, $\theta \in [0,2\pi]$, $\theta \in [0,\pi]$. *Thus* F *allows passage from cartesian to spherical coordinates in* $\mathbb{R}^3$.

1. Let $f(x,y,z)$ be an arbitrary differentiable function. Use the chain rule to compute

$$\frac{\partial}{\partial r} f(F(r,\theta,\phi)), \quad \frac{\partial}{\partial \theta} f(F(r,\theta,\phi)), \quad \frac{\partial}{\partial \phi} f(F(r,\theta,\phi)).$$

The formulas you get are of the form

$$\begin{aligned}
\frac{\partial}{\partial r} f(F(r,\theta,\phi)) &= a_{11}(r,\theta,\phi)\frac{\partial f}{\partial x}(F(r,\theta,\phi)) + a_{12}(r,\theta,\phi)\frac{\partial f}{\partial y}(F(r,\theta,\phi)) \\
&\quad + a_{13}(r,\theta,\phi)\frac{\partial f}{\partial z}(F(r,\theta,\phi)), \\
\frac{\partial}{\partial \theta} f(F(r,\theta,\phi)) &= a_{21}(r,\theta,\phi)\frac{\partial f}{\partial x}(F(r,\theta,\phi)) + a_{22}(r,\theta,\phi)\frac{\partial f}{\partial y}(F(r,\theta,\phi)) \\
&\quad + a_{23}(r,\theta,\phi)\frac{\partial f}{\partial z}(F(r,\theta,\phi)), \\
\frac{\partial}{\partial \phi} f(F(r,\theta,\phi)) &= a_{31}(r,\theta,\phi)\frac{\partial f}{\partial x}(F(r,\theta,\phi)) + a_{32}(r,\theta,\phi)\frac{\partial f}{\partial y}(F(r,\theta,\phi)) \\
&\quad + a_{33}(r,\theta,\phi)\frac{\partial f}{\partial z}(F(r,\theta,\phi)).
\end{aligned}$$

The coefficients $a_{ij}(r,\theta,\phi)$ *are the entries of the Jacobian matrix,* JF, *of the transformation* $(r,\theta,\phi) \mapsto F(r,\theta,\phi)$.

2. The matrix JF is invertible for $(r,\theta,\phi) \in \Sigma = \{(r,\theta,\phi) : r\sin\theta \neq 0\}$. Find the inverse of the matrix JF on Σ.

Writing the formulas above in matrix notation,

$$\begin{bmatrix} \frac{\partial f\circ F}{\partial r} \\ \frac{\partial f\circ F}{\partial \theta} \\ \frac{\partial f\circ F}{\partial \phi} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x}\circ F \\ \frac{\partial f}{\partial y}\circ F \\ \frac{\partial f}{\partial z}\circ F \end{bmatrix}$$

we have

$$\begin{bmatrix} \frac{\partial f}{\partial x}\circ F \\ \frac{\partial f}{\partial y}\circ F \\ \frac{\partial f}{\partial z}\circ F \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f\circ F}{\partial r} \\ \frac{\partial f\circ F}{\partial \theta} \\ \frac{\partial f\circ F}{\partial \phi} \end{bmatrix}$$

Let b_{ij} $(i,j = 1,2,3)$ *denote the entries of* JF^{-1}; *they are functions of* r, θ, *and* ϕ. *Let*

$$\begin{aligned}
L_1 &= b_{11}\frac{\partial}{\partial r} + b_{12}\frac{\partial}{\partial \theta} + b_{13}\frac{\partial}{\partial \phi} \\
L_2 &= b_{21}\frac{\partial}{\partial r} + b_{22}\frac{\partial}{\partial \theta} + b_{23}\frac{\partial}{\partial \phi} \\
L_3 &= b_{31}\frac{\partial}{\partial r} + b_{32}\frac{\partial}{\partial \theta} + b_{33}\frac{\partial}{\partial \phi}
\end{aligned}$$

3. Verify that

$$L_1(f(F(r,\theta,\phi)) = \frac{\partial f}{\partial x} f(F(r,\theta,\phi)), \quad L_2(f(F(r,\theta,\phi)) = \frac{\partial f}{\partial y}(F(r,\theta,\phi)),$$

$$L_3(f(F(r,\theta,\phi)) = \frac{\partial f}{\partial z}(F(r,\theta,\phi)).$$

4. For a function $v(r,\theta,\phi)$, compute

$$L_1(L_1(v)) + L_2(L_2(v)) + L_3(L_3(v)).$$

This gives a formula for the Laplacian in spherical coordinates.