1. Verify that the function

$$u(t,x) = e^{-t(\pi/3)^2} \sin \tfrac{\pi}{3} x - e^{-t(5\pi/3)^2} \sin \tfrac{5\pi}{3} x$$

(viewed as a function with domain $[0,\infty) \times [0,3]$) satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{for } (t,x) \in (0,\infty) \times (0,3) \\ u(t,0) = u(t,3) = 0 & \text{for } t > 0 \\ u(0,x) = \sin \tfrac{\pi}{3} x - \sin \tfrac{5\pi}{3} x & \text{if } x \in (0,3). \end{cases}$$

2. Let

$$u_0(x) = \sin \frac{\pi}{3} x - \sin \frac{5\pi}{3} x, \quad x \in [0,3].$$

Compute

$$\frac{2}{3} \int_0^3 u_0(y) \sin(\frac{k\pi}{3} y)\, dy.$$

3. Let P be the set of functions $f : [-1,1] \to \mathbb{R}$ of the form

$$f(x) = \sum_{k=0}^{10} a_k x^k. \qquad (\dagger)$$

This is a real vector space of dimension 11. Let $f_0 = 1,\ f_1 = x,\ f_2 = x^2, \ldots, f_{10} = x^{10}$. This is a basis of P.

a) Use Gram-Schmidt orthogonalization to find g_0, g_1, and g_2 of the form

$$g_k = \sum_{\ell=0}^{k} c_{k\ell} f_k$$

for some numbers $c_{k\ell}$ with the property that

$$\langle g_j, g_k \rangle = \delta_{jk}.$$

Here $\langle f, g \rangle$ is the inner product on P defined by setting

$$\langle f, g \rangle = \int_{-1}^{1} f(y) g(y)\, dy.$$

b) Let A be the subset of P consisting of 0 and those elements $(\dagger)$ involving only even powers of x, and let B be the set consisting of 0 and the elements of P involving only odd powers. Show that if $f \in A$ and $g \in B$, then

$$\langle f, g \rangle = 0.$$